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ISO-INTEGRAL EQUATIONS

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Abstract

This paper encompasses recent developments of iso-integral equations. They are given definitions for Volterra and Fredholm iso-integral equations of the first kind and of the second kind. They are considered the Adomian decomposition method, the modified decomposition method, the noise terms phenomenon, the iso-0differential method, the successive approximation method. It is proposed a method for conversion of a Volterra iso-integral equation of the first kind to a Volterra iso-integral equation of the second kind. It is given an existence result for Volterra iso-integral equations of the second kind. They are introduced resolvent kernels.

[0]Keywords and Phrases: iso-number, iso-differentail, iso-integral, existence, Adomian decomposition method, resolven kernel, modified decomposition method, successive approximations method

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1 Introduction

As it is well known, Isaac Newton had to develop the differential calculus, (jointly with Gottfried Leibniz), with particular reference to the historical definition of velocities as the time derivative of the coordinates, $v = dr/dt$, in order to write his celebrated equation $ma = F(t, r, v)$, where $a = dv/dt$ is the acceleration and $F(t, r, v)$ is the Newtonian force acting on the mass m. Being local, the differential calculus solely admitted the characterization of massive points. The differential calculus and the notion of massive points were adopted by Galileo Galilei and Albert Einstein for the formulation of their relativity, thus acquiring a fundamental role in 20th century sciences.

In his Ph. D. thesis of 1966 at the University of Turin, Italy, the Italian-American scientist Ruggero Maria Santilli pointed out that Newtonian forces are the most widely known in dynamics, including action-at-a-distance forces derivable derivable from a potential, thus representable with a Hamiltonian, and other forces that are not derivable from a potential or a Hamiltonian, since they are contact dissipative and non-conservative forces caused by the motion of the mass m within a physical medium. Santilli pointed out that, due to their lack of dimensions, massive points can solely experience action-at-a-distance Hamiltonian forces.

On this ground, Santilli initiated a long scientific journey for the generalization of Newton's equation into a form permitting the representation of the actual extended character of massive bodies whenever moving within physical media, as a condition to admit non-Hamiltonian forces. Being a theoretical physicist, Santilli had a number of severe physical conditions for the needed representation. One of them was the need for a representation of extended bodies and their non-Hamiltonian forces to be invariant over time as a condition to predict the same numerical values under the same conditions but at different times.

The resulting new calculus, today known as Santilli IsoDifferential Calculus, or IDC for short, stimulated a further layer of studies that finally signaled the achievement of mathematical and physical maturity. In particular, we note: the isotopies of Euclidean, Minkowskian, Riemannian and symplectic geometries; the isotopies of classical Hamiltonian mechanics, today known as the Hamilton-Santilli isomechanics, and the isotopies of quantum mechanics, today known as the isotopic branch of Hadronic mechanics.

Suppose that $\hat{T}_1$ is a positive constant and $\hat{T} : \mathbb{R} \times \mathbb{R} \longmapsto \mathcal{R}$ is a positive continuous function which is continuously-differentiable, $\hat{T}^1, \hat{T}^2 : \mathbb{R} \longmapsto \mathbb{R}$ are positive continuously-differentiable functions. Assume that the basic iso-unit in the field $\hat{\mathbb{R}}$ is $\frac{1}{\hat{T}_1}$, the basic iso-unit in the space of all continuously-

differentiable functions on $\mathbb{R}_x$ is $\frac{1}{\hat{T}^1(x)}$, the basic iso-unit in the space of all continuously-differentiable functions on $\mathbb{R}_y$ is $\frac{1}{\hat{T}^2(y)}$, and the basic iso-unit in the space of all continuously-differentiable functions on $\mathbb{R} \times \mathbb{R}$ is $\frac{1}{\hat{T}(x,y)}$.

The main aim of this paper is to be made a generalization of the theory of integral equations using the Santilli generalization of the basic uso-int. This theory will be called "Iso-Integral Equations".
If $\tilde{f}$, $\tilde{l}$ and $\tilde{g}$ are iso-functions of the first, second, third, fourth or fifth kind, then we define

$$\begin{aligned}
\tilde{f}(x)\hat{\times}\tilde{g}(x) &= \tilde{f}(x)\hat{T}^1(x)\tilde{g}(x), \\
\tilde{f}(y)\hat{\times}\tilde{g}(y) &= f(y)\hat{T}^2(y)g(y), \\
\tilde{f}(x)\hat{\times}\tilde{g}(y) &= \tilde{f}(x)\hat{T}(x,y)\tilde{g}(y), \\
\hat{a}\hat{\times}\tilde{f}(x) &= \hat{a}\hat{T}^1(x)\tilde{f}(x), \\
\hat{a}\hat{\times}\tilde{f}(y) &= \hat{a}\hat{T}^2(y)\tilde{f}(y), \\
\hat{a}\hat{\times}\tilde{h}(x,y) &= \hat{a}\hat{T}(x,y)\tilde{h}(x,y), \\
\tilde{f}(x)\hat{\times}\tilde{g}(y)\hat{\times}\tilde{l}(y) &= \tilde{f}(x)\hat{T}(x,y)\tilde{g}(y)\hat{T}(x,y)\tilde{l}(y).
\end{aligned}$$

2 Basic Definitions

Definition 2.1. *An iso-integral equation is the equation in which the unknown iso-function $\tilde{\phi}(x)$ of the first, second, third, fourth and fifth kind appears inside iso- integral sign.*

A standard type of iso-integral equation in $\tilde{\phi}(x)$ is of the following form

$$\tilde{\phi}(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\tilde{f}(x)}^{\tilde{g}(x)} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y}, \tag{2.1}$$

where $\tilde{f}, \tilde{g} : \mathcal{R} \longmapsto \mathcal{R}$ are the limits of iso-integration which can be iso-functions of the first, second, third, fourth and fifth kind, $\hat{\lambda}$ is an iso-constant parameter, and $\hat{K} : \mathcal{R} \times \mathcal{R} \longmapsto \mathcal{R}$ is a given iso-function of the first, second, third, fourth

and fifth kind of the variables x and y, $\tilde{u} : \mathcal{R} \longmapsto \mathcal{R}$ is a given iso-function of the first, second, third, fourth and fifth kind.

Definition 2.2. *The function $\tilde{K}(x,y)$ in (2.1) is called the kernel or the nucleus of the iso-integral equation.*

Example 2.1. Let $\hat{T}_1 = 3$, $\hat{T}(x) = x^2 + y^2$, $\hat{T}^1(x) = x^2 + 1$, $\hat{T}^2(y) = y^2 + 1$ $u(x) = x^3$, $f(x) = x$, $g(x) = x^2$, $K(x,y) = x + y$. Then

$$\hat{\phi}^{\wedge}(\hat{x}) = \hat{u}(\hat{x}) + \hat{4}\hat{\times}\hat{\int}_{f^{\wedge}(\hat{x})}^{g^{\wedge}(x)} K(\hat{x},\hat{y})\hat{\times}\hat{\phi}^{\wedge}(\hat{x})\hat{\times}d\hat{x}$$

is an iso-integral equation. This equation we can rewrite in the form

$$\phi(x) = \frac{x^3}{(x^2+1)^3} + 4\left(x^2+1\right)\int_x^{\frac{x^2}{(x^2+1)^2}} \left(\frac{x}{x^2+1} + \frac{y}{y^2+1}\right)\frac{1-y^2}{y^2+1}\phi(y)dy.$$

If $\hat{T}_1 \equiv \hat{T}(x,y) \equiv \hat{T}^1(x) \equiv \hat{T}^2(y)x \equiv 1$, then the considered equation takes the form

$$\phi(x) = x^3 + 4\int_x^{x^2} (x+y)\phi(y)dy.$$

In (2.1) the unknown function $\tilde{\phi}(x)$ appears the iso-integral sign. There are many other cases in which the unknown function $\tilde{\phi}(x)$ appears inside and outside the iso-integral sign. The functions $\tilde{u}(x)$ and $\tilde{K}(x,y)$ are given in advance. Note that the limits of integration $\tilde{f}(x)$ and $\tilde{g}(x)$ may be both constants, variables or mixed.

Definition 2.3. *If the limits of iso-integration are fixed, the equation (2.1) is called a Fredholm iso-integral equation given in the form*

$$\tilde{\phi}(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{b}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}d\hat{y}, \tag{2.2}$$

where $\hat{a}$ and $\hat{b}$ are iso-constants.

Example 2.2. Let $\hat{T}_1 = 2$, $a = 3$, $b = 5$, $\hat{T}(x,y) = x^2 + y^2 + 1$, $\hat{T}^1(x) = x^2 + 1$, $\hat{T}^2(y) = y^2 + y + 1$, $u(x) = x - 1$, $K(x,y) = 2x - 3y$. Then

$$\phi^{\wedge}(x) = u^{\vee}(x) + \hat{7}\hat{\times}\hat{\int}_{\hat{3}}^{\hat{5}} \hat{K}(\hat{x},\hat{y})\hat{\times}\hat{\phi}^{\wedge}(\hat{y})\hat{\times}d\hat{y}$$

is a Fredholm iso-integral equation.

This equation we can rewrite in the following form

$$\phi\left(\frac{x}{x^2+1}\right) = x^3 + x - 1$$

$$+7(x^2+1)\int_{\frac{3}{2}}^{\frac{5}{2}} \left(x^2+y^2+1\right) \times$$

$$\times \frac{(2xy^2 - 3yx^2 + 2xy + 2x - 3y)\left(-y^3 - y^2 + y + 1\right)}{(x^2+1)\left(y^2+y+1\right)^4}\phi(y)dy.$$

If $\hat{T}_1 \equiv \hat{T} \equiv \hat{T}^1 \equiv \hat{T}^2 \equiv 1$, then we have the integral equation

$$\phi(x) = x - 1 + 7\int_3^5 (2x - 3y)\phi(y)dy.$$

Definition 2.4. *If at least one limit is iso-variable, the equation is called Volterra iso-integral equation given in the form*

$$\tilde{\phi}(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y}.$$

Example 2.3. Let $\hat{T}_1 = 5$, $a = 1$, $u(x) = x$, $\hat{T}^1(x) = x^2+1$, $\hat{T}^2(y) = y^2+2y+3$, $\hat{T}(x,y) = x^2 + 2y^2 + 1$, $\lambda = 2$, $K(x,y) = x - 4y$. Then

$$\hat{\phi}(\hat{x}) = u^{\vee}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{1}}^{\hat{x}} \hat{K}(\hat{x},y)\hat{\times}\hat{\phi}^{\wedge}(\hat{y})\hat{\times}\hat{d}\hat{y}$$

is a Volterra iso-integral equation. This equation we can rewrite in the form

$$\phi\left(\frac{x}{x^2+1}\right) = x\left(x^2+1\right)^2$$

$$+ \quad 2\left(x^2+1\right)\int_{\frac{1}{5}}^{\frac{x}{x^2+1}} \left(-4yx^2 - 4y + x\right)\left(x^2+2y^2+1\right)\frac{3 - y - y^2}{\left(y^2+2y+3\right)^2}\phi(y)dy.$$

If $\hat{T}_1 \equiv \hat{T} \equiv \hat{T}^1 \equiv \hat{T}^2 \equiv 1$, then the considered equation takes the form

$$\phi(x) = x + 2\int_1^x (x - 4y)\phi(y)dy.$$

Definition 2.5. *If the unknown iso-function $\tilde{\phi}(x)$ appears only under the iso-integral sign of Fredholm iso-integral equation or Volterra iso-integral equation, the iso-integral equation is called a first kind Fredholm or Volterra iso-integral equation, respectively.*

Example 2.4. Let

$$U_x = \{x \in \mathbb{R} : x \geq 0\}, \quad U_y = \{y \in \mathbb{R} : y \geq 0\},$$

$\hat{T}_1 = 2$, $\hat{T}(x,y) = x + y + 1$, $\hat{T}^1(x) = x + 1$, $\hat{T}^2(y) = y + 1$, $u(x) = 2x - 3$, $K(x,y) = x + y$, $\lambda = 1$, $a = 4$, $b = 6$, $(x,y) \in U_x \times U_y$. Then the equation

$$\hat{u}^{\wedge}(\hat{x}) = \hat{1}\hat{\times}\hat{\int}_{\hat{4}}^{\hat{6}} \hat{K}(\hat{x},\hat{y})\hat{\times}\hat{\phi}^{\wedge}(y)\hat{\times}\hat{d}\hat{y}$$

is a first-kind Fredholm iso-integral equation. This equation we can rewrite in the following form

$$2x - 3 = (x+1)\int_2^3 \frac{x + 2xy + y}{(y+_1)^3}\phi\left(\frac{y}{y+1}\right)dy.$$

If $\hat{T}_1 \equiv \hat{T} \equiv \hat{T}^1 \equiv \hat{T}^2 \equiv 1$, the considered equation takes the form

$$2x - 3 = \int_4^6 (x + y)\phi(y)dy.$$

Example 2.5. Let

$$U_x = \{x \in \mathbb{R} : x \geq 0\}, \quad U_y = \{y \in \mathbb{R} : y \geq 0\},$$

$\hat{T}_1 = 3$, $\hat{T}(x,y) = x + y + 2$, $\hat{T}^1(x) = x + 2$, $\hat{T}^2(y) = y + 3$, $a = 4$, $\lambda = 2$, $K(x,y) = 2x - 4y + 1$, $u(x) = 3x - 2$, $(x,y) \in U_x \times U_y$. Then the equation

$$\phi^{\vee}(x) = \hat{u}^{\wedge}(x) + \hat{2}\hat{\times}\hat{\int}_{\hat{4}}^{\hat{x}} K(x,\hat{y})\hat{\times}\phi^{\wedge}(y)\hat{\times}\hat{d}\hat{y}$$

is a first-kind Volterra iso-integral equation. We can rewrite it in the form

$$\begin{aligned}\phi\left(\frac{x}{x+2}\right) &= \frac{3x^2 + 6x - 2}{x+2} \\ &\quad +6(x+2)\int_{\frac{4}{3}}^{\frac{x}{x+2}} \frac{6x + 2xy - 5y - 3}{(y+3)^2}(x + y + 2)^2\phi\left(y^2 + 3y\right)dy.\end{aligned}$$

If $\hat{T}_1 \equiv \hat{T} \equiv \hat{T}^1 \equiv \hat{T}^2 \equiv 1$, the considered equation takes the form

$$\phi(x) = 3x - 2 + 2\int_4^x (2x - 4y + 1)\phi(y)dy.$$

Definition 2.6. *If the unknown iso-function $\tilde{\phi}(x)$ appears inside and outside the iso-integral sign of the Fredholm iso-integral equation or Volterra iso-integral equation, then it is called a second kind Fredholm iso-integral equation or Volterra iso-integral equation, respectively.*

Definition 2.7. *If in the Fredholm iso-integral equation or Volterra iso-integral equation the iso-function $\tilde{u}(x)$ is identically zero, the resulting equations*

$$\tilde{\phi}(x) = \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{b}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{d}\hat{y}$$

or

$$\tilde{\phi}(x) = \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y}$$

is called homogeneous Fredholm iso-integral equation or homogeneous Volterra iso-integral equation, respectively.

Definition 2.8. *Any equation that includes iso-integrals and iso-derivatives of the unknown iso-function $\tilde{\phi}(x)$ is called iso-integral-iso-differential equation.*

The Fredholm iso-integral-iso-differential equation is of the form

$$\tilde{\phi}^{\circledast^k}(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{b}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y}.$$

However, the Volterra iso-integral-iso-differential equation is of the form

$$\tilde{\phi}^{\circledast^k}(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y}.$$

Definition 2.9. *If the unknown iso-function $\tilde{\phi}(x)$ inside iso-integral sign is one, the iso-integral equation or the iso-integral equation or the iso-integral-iso-differential equation is called linear.*

Definition 2.10. *If the equation contains nonlinear function of the unknown iso-function $\tilde{\phi}(x)$, the iso-integral equation or the iso-integral-iso-differential equation is called nonlinear.*

3 The Adomian Decomposition Method

The Adomian decomposition method (ADM) consists of decomposing the unknown iso-function $\tilde{\phi}(x)$ of any Volterra iso-integral equation into a sum of infinite number of components by the series

$$\tilde{\phi}(x) = \sum_{n=0}^{\infty} \tilde{\phi}_n(x) \tag{3.1}$$

or equivalently

$$\tilde{\phi}(x) = \tilde{\phi}_0(x) + \tilde{\phi}_1(x) + \tilde{\phi}_2(x) + \cdots,$$

where the components $\tilde{\phi}_l(x)$, $l \in \mathcal{N}_0$, are to be determined in a recursive manner.

To establish the recursive relation, we substitute (3.1) into the Volterra isointegral equation

$$\tilde{\phi}(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\int_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\hat{\phi}(y)\hat{\times}d\hat{y}. \tag{3.2}$$

We obtain

$$\sum_{n=0}^{\infty}\tilde{\phi}_n(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\int_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\sum_{n=0}^{\infty}\tilde{\phi}_n(y)\hat{\times}d\hat{y},$$

or equivalently,

$$\begin{aligned}\tilde{\phi}_0(x) + \tilde{\phi}_1(x) + \tilde{\phi}_2(x) + \cdots \;=\; & \tilde{u}(x) + \hat{\lambda}\hat{\times}\int_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}_0(y)\hat{\times}d\hat{y} \\ & + \hat{\lambda}\hat{\times}\int_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}_1(y)\hat{\times}d\hat{y} \\ & + \hat{\lambda}\hat{\times}\int_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}_2(y)\hat{\times}d\hat{y} \\ & + \cdots.\end{aligned}$$

We set

$$\begin{aligned}\tilde{\phi}_0(x) &= \tilde{u}(x), \\ \tilde{\phi}_1(x) &= \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}}\tilde{K}(x,y)\tilde{\phi}_0(y)\hat{\times}d\hat{y}, \\ \tilde{\phi}_2(x) &= \hat{\lambda}\hat{\int}_{\hat{a}}^{\hat{x}}\tilde{K}(x,y)\hat{\times}\tilde{\phi}_1(y)\hat{\times}d\hat{y},\end{aligned}$$

and so on for the other components, or equivalently,

$$\begin{aligned}\tilde{\phi}_0(x) &= \tilde{u}(x), \\ \tilde{\phi}_n(x) &= \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}}\tilde{K}(x,y)\hat{\times}\tilde{\phi}_{n-1}(y)\hat{\times}d\hat{y}, \quad n \in \mathcal{N}.\end{aligned} \tag{3.3}$$

In view of (3.3), the components $\tilde{\phi}_0(x)$, $\tilde{\phi}_1(x)$, $\tilde{\phi}_2(x)$, ... are completely determined. The solution $\tilde{\phi}(x)$ of (3.2) in a series form is obtained by using the series (3.1).

In other words, the Adomian decomposition method converts the generalized Volterra integral equation into a determination of computable components. Note that if an exact solution exists for (3.2), then the obtained series converges to that solution. However, for some equations, when a closed form solution is not obtainable, a number of terms of (3.1) can be used for numerical purposes. The question for convergence of $\tilde{\phi}_n(x)$ will be discussed in the last section of this chapter.

4 The Modified Decomposition Method

For many cases, the function $\tilde{u}(x)$ can be represented as the sum of two partial iso-functions, namely, $\tilde{u}_1(x)$ and $\tilde{u}_2(x)$, i.e.,

$$\tilde{u}(x) = \tilde{u}_1(x) + \tilde{u}_2(x).$$

The modified decomposition method(MDM) introduces the modified recurrence relation.

$$\begin{aligned}
\tilde{\phi}_0(x) &= \tilde{u}_1(x), \\
\tilde{\phi}_1(x) &= \tilde{u}_2(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}_0(y)\hat{\times}d\hat{y}, \\
\tilde{\phi}_n(x) &= \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}_{n-1}(y)\hat{\times}d\hat{y}, \quad n \in \mathcal{N}\backslash\{1\}.
\end{aligned} \tag{4.1}$$

The difference between the standard recurrence (3.3) and the modified recurrence relation (4.1) is in the information of the first two components $\tilde{\phi}_0(x)$ and $\tilde{\phi}_1(x)$. The other components $\tilde{\phi}_n(x)$, $n \geq 2$, remain the same in the two recurrence formulas. The success of this modification depends only on the proper choice of $\tilde{u}_1(x)$ and $\tilde{u}_2(x)$. A rule for such choice of $\tilde{u}_1(x)$ and $\tilde{u}_2(x)$ could not be found yet. If $\tilde{u}(x)$ consists of one term only, the standard decomposition method can be used in this case.

5 The Noise Terms Phenomenon

The noise terms are defined as the identical terms with opposite signs which can appear in the components $\tilde{\phi}_0(x)$ and $\tilde{\phi}_1(x)$ and in the other components as well. Noise terms may appear if the exact solutions of the considered equation is part of the zeroth component $\tilde{\phi}_0(x)$. By cancelling the noise terms between $\tilde{\phi}_0(x)$ and $\tilde{\phi}_1(x)$, the remaining non-cancelled terms of $\tilde{\phi}_0(x)$ may give the exact solution of the given equation. Verification that the remaining non-cancelled terms satisfy the integral equation is necessary and essential. Note that the

noise terms appear for specific cases of inhomogeneous integral equations. Homogeneous integral equations do not give rise to noise terms. The question for convergence of $\tilde{\phi}_n(x)$ will be discussed in the last section of this chapter.

6 Iso-Differential Method

Some Volterra iso-integral equations reducing them to IVPs for some iso-differential equations. This method is known as Iso-Differential Equation Method(Iso-DEM). Consider the Volterra iso-integral equation

$$\tilde{\phi}(\hat{x}) = \tilde{u}(\hat{x}) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}d\hat{y}.$$

Then after we iso-differentiate the above equation we get the following IVP

$$\left(\tilde{\phi}(\hat{x})\right)^{\circledast} = (\tilde{u}(\hat{x}))^{\circledast} + \hat{\lambda}\tilde{K}(x,\hat{x})\hat{\times}\tilde{\phi}(\hat{x}),$$

$$\tilde{\phi}(\hat{a}) = \tilde{u}(\hat{a}).$$

7 The Successive Approximations Method

The successive approximations method(SAM), also called the Picard iteration method, provides a scheme that can be used for solving initial value problems or generalized integral equations. We set

$$\tilde{\phi}_1(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}}\tilde{K}(x,y)\tilde{\phi}_0(y)\hat{\times}d\hat{y},$$

$$\tilde{\phi}_2(x) = \tilde{u}(x) + \hat{\lambda}\hat{\int}_{\hat{a}}^{\hat{x}}\tilde{K}(x,y)\hat{\times}\tilde{\phi}_1(y)\hat{\times}d\hat{y},$$

and so on for the other components, or equivalently,

$$\hat{\phi}_n(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}}\tilde{K}(x,y)\hat{\times}\tilde{\phi}_{n-1}(y)\hat{\times}d\hat{y}, \quad n \in \mathcal{N}. \tag{7.1}$$

In view of (7.1), the components $\tilde{\phi}_0(x)$, $\tilde{\phi}_1(x)$, $\tilde{\phi}_2(x)$, ... are completely determined.

8 Conversion of a Volterra Iso-Integral Equation of the First Kind to a Volterra Iso-Integral Equation of the Second Kind

Consider a Volterra iso-integral equation of the first kind

$$\tilde{u}(\hat{x}) = \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y}.$$

We iso-differentiate this equation and we get

$$\begin{aligned}(\tilde{u}(x))^{\circledast} &= \hat{\lambda}\hat{\times}\tilde{K}(x,\hat{x})\hat{\times}\tilde{\phi}(\hat{x}) \\ &\quad +\hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \left(\tilde{K}(x,y)\right)^{\circledast}_{\hat{x}} \hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y},\end{aligned}$$

whereupon

$$\begin{aligned}\tilde{\phi}(\hat{x}) &= \left((\tilde{u}(\hat{x}))^{\circledast}\right) \nearrow \left(\hat{\lambda}\hat{\times}\tilde{K}(x,\hat{x})\right) \\ &\quad - \left(\hat{\lambda} \nearrow \left(\hat{\lambda}\hat{\times}\tilde{K}(x,\hat{x})\right)\right) \hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \left(\tilde{K}(x,y)\right)^{\circledast}_{\hat{x}} \hat{\times}\tilde{\phi}(y)\hat{\times}\hat{d}\hat{y},\end{aligned}$$

which is a Volterra iso-integral equation of the second kind. Hence, all methods as Adomian Decomposition Method, Modified Decomposition Method, Noise Term Phenomenon, Successive Approximations Method can be applied for Volterra iso-integral equations of the first kind.

9 Existence of Solutions of Volterra Iso-Integral Equations of the Second Kind

Theorem 9.1. *Let $\tilde{K}(x,y)$ be a real-valued continuous iso-function defined on $a \leq x, y \leq b$, $\tilde{u}(x)$ be a real-valued continuous iso-function defined on $a \leq x \leq b$.*

Let also,

$$|\tilde{u}(x)| \leq M \quad for \quad all \quad x \in [a,b],$$

$$|\tilde{K}(x,y)| \leq N \quad for \quad all \quad x,y \in [a,b],$$

$$\left|\frac{\hat{T}^2(y)-\left(y\hat{T}^2(y)\right)'}{\hat{T}^2(y)}\right| \leq M \quad for \quad all \quad y \in [a,b],$$

$$\left|\frac{\hat{T}^1(x)-\left(y\hat{T}^1(x)\right)'}{\hat{T}^1(x)}\right| \leq M \quad for \quad all \quad x \in [a,b],$$

$$0 \leq \hat{T}^1(x) \leq M, \quad 0 \leq \hat{T}^2(y) \leq M \quad for \quad all \quad x,y \in [a,b].$$

Then for $|\lambda| < \infty$ *the Volterra iso-integral equation of the second kind*

$$\tilde{\phi}(x) = \tilde{u}(x) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,y)\hat{\times}\tilde{\phi}(y)\hat{\times}\hat{dy}, \quad a \leq x \leq b, \tag{9.1}$$

has unique continuous solution $\tilde{\phi}(x)$ *defined on* $[a,b]$.

Because we can convert Volterra iso-integral equations of the first kind to Volterra iso-integral equations of the second, under the conditions of the above theorem it follows that the Volterra iso-integral equation of the first kind has unique continuous solution.

10 Resolvent Kernels

We define

$$\tilde{K}^{(1)}(x,y) \;=\; \hat{\int}_{\hat{a}}^{\hat{x}} \tilde{K}(x,z)\hat{\times}\left(\tilde{K}(z,y)\right)^{\hat{n-1}}\hat{\times}\hat{dz}, \quad n \in \mathbb{N}.$$

We set

$$\tilde{\Gamma}(x,y,\hat{\lambda}) = \tilde{K}(x,y) + \hat{\lambda}\hat{\times}\tilde{K}^{(2)}(x,y) + \hat{\lambda}^{\hat{2}}\hat{\times}\tilde{K}^3(x,y) + \cdots.$$

Definition 10.1. *The function* $\tilde{\Gamma}(x,y,\hat{\lambda})$ *is called the resolvent kernel of the kernel* $\tilde{K}(x,y)$.

Theorem 10.1. *Let* $\tilde{K}(x,y)$ *be a real-valued continuous iso-function defined on* $a \leq x,y \leq b$, $\tilde{u}(x)$ *be a real-valued continuous iso-function defined on* $a \leq x \leq b$.

Let also,

$$|\tilde{u}(x)| \leq M \quad for \quad all \quad x \in [a,b],$$

$$|\tilde{K}(x,y)| \leq N \quad for \quad all \quad x,y \in [a,b],$$

$$\left|\frac{\hat{T}^2(y)-\left(y\hat{T}^2(y)\right)'}{\hat{T}^2(y)}\right| \leq M \quad for \quad all \quad y \in [a,b],$$

$$\left|\frac{\hat{T}^1(x)-\left(y\hat{T}^1(x)\right)'}{\hat{T}^1(x)}\right| \leq M \quad for \quad all \quad x \in [a,b],$$

$$0 \leq \hat{T}^1(x) \leq M, \quad 0 \leq \hat{T}^2(y) \leq M \quad for \quad all \quad x,y \in [a,b].$$

Then for $|\lambda| < \infty$ *the resolvent kernel satisfies the iso-integral equation*

$$\tilde{\Gamma}(x,y,\hat{\lambda}) = \tilde{K}(x,y) + \hat{\lambda}\hat{\times}\hat{\int}_{\hat{y}}^{\hat{x}} \tilde{K}(x,z)\hat{\times}\Gamma(x,y,\hat{\lambda})\hat{\times}\hat{d}\hat{z}.$$

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EXISTENCE AND UNIQUENESS OF THE SOLUTIONS FOR A CLASS ORDINARY ISO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper is investigated a class ordinary iso-differential equations. They are formulated and proved existence and uniqueness results for the considered class ordinary iso-differential equations.

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1 introduction

Let $(x_0, y_0) \in \mathbb{R}^2$, D is a domain in $\mathbb{R}^2$ containing the point (x_0, y_0), J is an interval in $\mathbb{R}$ containing x_0, $\hat{T}(x) \in C^1(J)$, $\hat{T}(x) > 0$ for every $x \in J$.

We begin to develop the theory of existence and uniqueness of solutions of the initial value problem

$$(1') \qquad \Big(\hat{y}\hat{}(\hat{x})\Big)^{\circledast} = \hat{f}\hat{}(\hat{x}, \hat{y}\hat{}(\hat{x})), \qquad x \in J,$$

$$(2) \qquad y(x_0) = y_0,$$

where f will be assumed to be continuous in the domain D.

The equation (1') can be rewritten in the following form

$$\frac{y'(x)\hat{T}(x) - y(x)\hat{T}'(x)}{\hat{T}(x)\Big(\hat{T}(x) - x\hat{T}'(x)\Big)} = \frac{f(x, y(x))}{\hat{T}(x)}, \qquad x \in J,$$

or

$$y'(x)\hat{T}(x) - y(x)\hat{T}'(x) = f(x, y(x))\Big(\hat{T}(x) - x\hat{T}'(x)\Big), \qquad x \in J,$$

or

$$(1) \qquad y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}, \qquad x \in J.$$

Definition 1.1. *We will say that a function $y(x)$ is a solution to the initial value problem (1), (2) if*

1. $y(x_0) = y_0$,
2. $y'(x)$ *exists for all* $x \in J$,
3. *for all* $x \in J$ *the points* $(x, y(x)) \in D$,
4. $y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$ *for all* $x \in J$.

If $f(x, y(x))$ is not continuous, then the nature of the solutions of (1) is quite arbitrary. For example, let

$$f(x, y(x)) = \frac{4(y(x) - 2)}{x(1 - x)} - \frac{y(x)}{1 - x}, \qquad \hat{T}(x) = e^x,$$

and $(x_0, y(x_0)) = (0, 0)$. Then the equation (1) admits the representation

$$y'(x) = y(x) + \Big(\frac{4(y(x) - 2)}{x(1 - x)} - \frac{y(x)}{1 - x}\Big)(1 - x)$$

$$= y(x) + 4\frac{y(x) - 2}{x} - y(x)$$

$$= \frac{4}{x}(y(x) - 2),$$

its general solution is

$$(3) \qquad y(x) = 2 + Cx^4,$$

where C is a constant. From here, we conclude that

$$y(0) = 2 \neq 0,$$

therefore the considered initial value problem has no any solution. If we take $(x_0, y(x_0)) = (0, 2)$, then every function (2) will be a solution of the considered initial value problem.

We shall need the following result to prove existence, uniqueness, and several other properties of the solutions of the initial value problem (1), (2).

Theorem 1.1. *Let $f(x, y(x))$ be continuous function in the domain D, then any solution of the initial value problem* (1), (2) *is also a solution of the integral equation*

$$(4) \qquad y(x) = y_0 + \int_{x_0}^{x} \Big(y(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

and conversely.

Proof. An integration of the equation (1) yields

$$y(x) - y(x_0) = \int_{x_0}^{x} \Big(y(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

Conversely, if $y(x)$ is any solution of (4), then

$$y(x_0) = y_0,$$

and since $f(x, y(x))$ is a continuous function in D and $\hat{T}$ is a continuous function in J, then $y(x)$ is a continuous function in J and we can differentiate (4), from where we find

$$y'(x) = y(x) \frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x)) \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}.$$

□

We shall solve the integral equation (4) by using the method of successive approximations due to Picard. For this reason, let $y_0(x)$ be any continuous function, we often take $y_0(x) \equiv y_0$, which we will suppose to be initial approximation of the unknown solution of (4), then we define $y_1(x)$ as follows

$$y_1(x) = y_0 + \int_{x_0}^{x} \Big(y_0(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_0(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

We pick this $y_1(x)$ as our next approximation and substitute this for $y(x)$ in the right side of (4) and call it $y_2(x)$,

$$y_2(x) = y_0 + \int_{x_0}^{x} \Big(y_1(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

Continuing in this way, the $(m+1)$st approximation $y_{m+1}(x)$ is obtained from $y_m(x)$ by means of the relation

$$(5) \qquad y_{m+1}(x) = y_0 + \int_{x_0}^{x} \Big(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt, \qquad m = 0, 1, 2, \ldots.$$

If the sequence $\{y_m(x)\}_{m=1}^{\infty}$ converges uniformly to a continuous function $y(x)$ in the interval J and for all $x \in J$ the points $(x, y_m(x)) \in D$, then we may pass to the limit in both sides of (5), to obtain

$$y(x) = \lim_{m \to \infty} y_{m+1}(x)$$

$$= y_0 + \lim_{m \to \infty} \int_{x_0}^{x} \Big(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y_0 + \int_{x_0}^{x} \Big(y(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt,$$

so that $y(x)$ is the desired solution.

Example 1.1. Let $J = [-1, 1]$, $D = \{(x, y) : \quad -1 \le x \le 1, \quad y \in \mathbb{R}\}$, $\hat{T}|(x) = e^x$, $f(x, y(x)) = y(x)$, $y(0) = 1$. We will find the first four approximation.

We pick

$$y_0(x) \equiv y_0 = 1,$$

$$y_1(x) = 1 + \int_0^x (2 - t) y_0(t) dt$$

$$= 1 + \int_0^x (2 - t) dt$$

$$= 1 + 2x - \tfrac{x^2}{2},$$

$$y_2(x) = 1 + \int_0^1 (2 - t) y_1(t) dt$$

$$= 1 + \int_0^x (2 - t) \Big(1 + 2t - \tfrac{t^2}{2} \Big) dt$$

$$= 1 + \int_0^x \Big(2 + 3t - 3t^2 + \tfrac{t^3}{2} \Big) dt$$

$$= 1 + 2x + \tfrac{3}{2} x^2 - x^3 + \tfrac{x^4}{8},$$

$$y_3(x) = 1 + \int_0^x (2-t)\Big(1 + 2t + \tfrac{3}{2}t^2 - t^3 + \tfrac{t^4}{8}\Big)dt$$

$$= 1 + \int_0^x \Big(2 + 3t + t^2 - \tfrac{7}{2}t^3 + \tfrac{5}{4}t^4 - \tfrac{t^5}{8}\Big)dt$$

$$= 1 + 2x + \tfrac{3}{2}x^2 + \tfrac{x^3}{3} - \tfrac{7}{8}x^4 + \tfrac{1}{4}x^5 - \tfrac{x^6}{48}.$$

Example 1.2. Let $J = \mathbb{R}$, $D = \mathbb{R}^2$, $f(x, y(x)) = -\frac{2y(x)}{1-x}$, $\hat{T}(x) = e^x$, $y(0) = 1$. Then the equation (1) we can represent in the form

$$y'(x) = y(x) - 2\tfrac{y(x)}{1-x}(1-x)$$

$$= y(x) - 2y(x)$$

$$= -y(x).$$

Then

$$y_0(x) \equiv y_0 = 1,$$

$$y_1(x) = 1 - \int_0^x y_0(t)dt$$

$$= 1 - x,$$

$$y_2(x) = 1 - \int_0^x y_1(t)dt$$

$$= 1 - \int_0^x (1-t)dt$$

$$= 1 - x + \tfrac{x^2}{2},$$

$$y_3(x) = 1 - \int_0^x \Big(1 - t + \tfrac{t^2}{2}\Big)dt$$

$$= 1 - x + \tfrac{x^2}{2} - \tfrac{x^3}{3!}.$$

We assume that

$$y_m(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots + (-1)^m \frac{x^m}{m!}.$$

Then

$$y_{m+1}(x) = 1 - \int_0^x \Big(1 - t + \tfrac{t^2}{2} - \tfrac{t^3}{3!} + \cdots + (-1)^m \tfrac{T^m}{m!}\Big)dt$$

$$= 1 - x + \tfrac{x^2}{2} - \tfrac{x^3}{3!} + \cdots + (-1)^{m+1} \tfrac{x^{m+1}}{(m+1)!}.$$

Therefore

$$y_m(x) = \sum_{i=0}^{m} (-1)^i \frac{x^i}{i!}, \qquad m = 0, 1, 2, \ldots.$$

From here,

$$\lim_{m\longrightarrow\infty} y_m(x) = e^{-x}.$$

Consequently

$$y(x) = e^{-x}$$

is a solution to the considered problem.

2 Existence and Uniqueness Results

Below we will suppose that a and b are positive real numbers. Let P be positive real number such that

$$\frac{|\hat{T}'(x)|}{\hat{T}(x)} \leq P, \qquad \frac{|\hat{T}(x) - x\hat{T}'(x)|}{\hat{T}(x)} \leq P \qquad \text{for} \qquad \forall x \in [x_0 - a, x_0 + a].$$

Theorem 2.1. *Let the following conditions be satisfied*

(i) *$f(x,y)$ is continuous in the closed rectangle $\overline{S}: |x - x_0| \leq a$, $|y - y_0| \leq b$ and hence there exists a $M > 0$ such that $|f(x,y)| \leq M$ for all $(x,y) \in \overline{S}$,*

(ii) *$f(x,y)$ satisfies a uniform Lipschitz condition*

$$|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|$$

for all (x,y_1), (x,y_2) in the closed rectangle $\overline{S}$,

(iii) *$y_0(x)$ is continuous in $|x - x_0| \leq a$, and $|y_0(x) - y_0| \leq b$.*

Then the sequence $\{y_m(x)\}_{m=1}^{\infty}$ generated by Picard iterative scheme (5) converges to the unique solution $y(x)$ of the initial value problem (1), (2). This solution is valid in the interval $J_h: |x - x_0| \leq h$, where $h = \min\left\{a, \frac{b}{P(b+|y_0|+M)}\right\}$. Further, for all $x \in J_h$ the following error estimate holds

(6) $$|y(x) - y_m(x)| \leq Ne^{(P+PL)h} \min\left\{1, \frac{((P+PL)h)^m}{m!}\right\}, \qquad m = 0, 1, 2, \ldots,$$

where

$$\max_{x\in J_h} |y_1(x) - y_0(x)| \leq N.$$

Remark 2.1. *This Theorem is called a local existence theorem since it guarantees a solution only in the neighborhood of the point (x_0, y_0).*

Proof. We will show that the successive approximations $y_m(x)$ defined by (5) exist as continuous function in J_h and $(x, y_m(x)) \in \overline{S}$ for all $x \in J_h$. Since $y_0(x)$ is a continuous function for all x such that $|x - x_0| \leq a$, the function $F_0(x) = f(x, y_0(x))$ is continuous function in J_h, and hence $y_1(x)$ is continuous in J_h.

Also,

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} \left(y_0(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_0(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_{x_0}^{x} \left(|y_0(t)| \frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_0(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_{x_0}^{x} \Big((b + |y_0|)P + MP \Big) dt \right|$$

$$= P(b + |y_0| + M)|x - x_0|$$

$$\leq P(b + |y_0| + M)h$$

$$\leq b.$$

Assuming that the assertion is true for $y_m(x)$, $m \geq 1$, then it is sufficient to prove that it is also true for $y_{m+1}(x)$. For this, since $y_m(x)$ is continuous in J_h, the function $F_m(x) = f(x, y_m(x))$ is also continuous function in J_h. Moreover,

$$|y_{m+1}(x) - y_0| = \left| \int_{x_0}^{x} \left(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_{x_0}^{x} \left(|y_m(t)| \frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_{x_0}^{x} \Big((b + |y_0|)P + MP \Big) dt \right|$$

$$\leq P(b + |y_0| + M)|x - x_0|$$

$$\leq P(b + |y_0| + M)h$$

$$\leq b.$$

Now we will prove that the sequence $\{y_m(x)\}_{m=1}^{\infty}$ converges uniformly in J_h. Since $y_1(x)$ and $y_0(x)$ are continuous in J_h, there exists a constant $N > 0$ such that

$$|y_1(x) - y_0(x)| \leq N \qquad \text{for} \qquad \forall x \in J_h.$$

Also, for every $x \in J_h$, we have

$$|y_2(x) - y_1(x)| = \left|\int_{x_0}^{x}\left((y_1(t) - y_0(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_1(t)) - f(t, y_0(t)))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_1(t) - y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_0(t))|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_1(t) - y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + L|y_1(t) - y_0(t)|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}(NP + LNP)dt\right|$$

$$= NP(1 + L)|x - x_0|.$$

Supposing that

$$(7) \qquad |y_m(x) - y_{m-1}(x)| \leq N\frac{\Big((P+LP)|x-x_0|\Big)^{m-1}}{(m-1)!}, \qquad x \in J_h,$$

for some $m \in \mathbb{N}$.

We will prove that

$$|y_{m+1}(x) - y_m(x)| \leq N\frac{\Big((P + LP)|x - x_0|\Big)^{m}}{m!}, \qquad x \in J_h.$$

Really,

$$|y_{m+1}(x) - y_m(x)|$$

$$= \left|\int_{x_0}^{x}\left((y_m(t) - y_{m-1}(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_m(t)) - f(t, y_{m-1}(t)))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t)) - f(t, y_{m-1}(t))|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + L|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}(P + PL)|y_m(t) - y_{m-1}(t)|dt\right|$$

$$\leq N(P + PL)^{m+1}\left|\int_{x_0}^{x}\frac{(t-x_0)^m}{m!}dt\right|$$

$$= N(P + PL)^{m+1}\frac{|x-x_0|^{m+1}}{(m+1)!}.$$

Thus inequality (7) is true for all $m \in \mathbb{N}$.

Next, since

$$N\sum_{m=1}^{\infty}\frac{\Big((P+PL)|x-x_0|\Big)^{m-1}}{(m-1)!} \le N\sum_{m=0}^{\infty}\frac{\Big((P+PL)h\Big)^{m}}{m!}$$

$$= Ne^{(P+PL)h} < \infty,$$

we have that the series

$$y_0(x) + \sum_{m=1}^{\infty}(y_m(x) - y_{m-1}(x))$$

converges absolutely and uniformly in the interval J_h, and hence its partial sums

$$y_1(x), y_2(x), \ldots, y_m(x), \ldots$$

converge to a continuous function in this interval, i.e.,

$$y(x) = \lim_{m\longrightarrow\infty} y_m(x).$$

As we have seen above we have that $y(x)$ is a solution to the problem (1), (2).

To prove that $y(x)$ is the only solution, we assume that $z(x)$ is also a solution to the initial value problem (1), (2) which exists in the interval J_h and $(x, z(x)) \in \overline{S}$ for all $x \in J_h$. The hypothesis (ii) is applicable and we have

$$|y(x) - z(x)| \le \left|\int_{x_0}^{x}\Big(|y(t) - z(t)|\tfrac{\hat{T}'(t)}{\hat{T}(t)} + |f(t, y(t)) - f(t, z(t))|\tfrac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\Big)dt\right|$$

$$\le \left|\int_{x_0}^{x}\Big(P|y(t) - z(t)| + LP|y(t) - z(t)|\Big)dt\right|$$

$$= (P + LP)\left|\int_{x_0}^{x}|y(t) - z(t)|dt\right|, \qquad x \in J_h.$$

Consequently

$$|y(x) - z(x)| = 0$$

for all $x \in J_h$.

Finally, we will obtain the error bound (6).

For $n > m$ the inequality (7) gives

$$|y_n(x) - y_m(x)| = |y_n(x) - y_{n-1}(x) + y_{n-1}(x) - y_{n-2}(x) + \cdots + y_{m+1}(x) - y_m(x)|$$

$$\leq \sum_{k=m}^{n-1} |y_{k+1}(x) - y_k(x)|$$

$$\leq N \sum_{k=m}^{n-1} \frac{\Big((P+LP)|x-x_0|\Big)^k}{k!}$$

$$\leq N \sum_{k=m}^{n-1} \frac{\Big((P+PL)h\Big)^k}{k!} \tag{8}$$

$$= N\Big((P+PL)h\Big)^m \sum_{k=0}^{n--m-1} \frac{\Big((P+PL)h\Big)^k}{(m+k)!} \qquad \left(\frac{1}{(m+k)!} \leq \frac{1}{m!k!}\right)$$

$$\leq N \frac{\Big((P+PL)h\Big)^m}{m!} \sum_{k=0}^{n-m-1} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$\leq N \frac{\Big((P+PL)h\Big)^m}{m!} e^{(P+PL)h},$$

and hence as $n \longrightarrow \infty$, we get

$$|y(x) - y_m(x)| \leq N \frac{\Big((P+PL)h\Big)^m}{m!} e^{(P+PL)h}$$

in J_h.

The inequality (8) provides

$$|y_n(x) - y_m(x)| \leq N \sum_{k=m}^{n-1} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$\leq N \sum_{k=0}^{\infty} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$= N e^{(P+PL)h},$$

and as $n \longrightarrow \infty$, we find

$$|y(x) - y_m(x)| \leq N e^{(P+PL)h}$$

in J_h. □

Example 2.1. Let $f(x,y) = -(x + y(x))$, $\hat{T}(x) = e^x$, $y(0) = 1$. Then the equation (1) we can rewrite in the following form

$$y'(x) = y(x) - x(1-x) - y(x)(1-x)$$

$$= xy(x) + x(x-1).$$

Its general solution is

$$y(x) = e^x\Big(C + \int x(x-1)e^{-x}dx\Big)$$

$$= e^x\Big(C - (x^2-x)e^{-x} + \int(2x-1)e^{-x}dx\Big)$$

$$= e^x\Big(C - (x^2-x)e^{-x} - (2x-1)e^{-x} + 2\int e^{-x}dx\Big)$$

$$= e^x\Big(C - (x^2+x-1)e^{-x} - 2e^{-x}\Big)$$

$$= Ce^x - x^2 - x - 1.$$

From here,

$$y(0) = C - 1 = 1,$$

therefore

$$C = 2.$$

Consequently

$$y(x) = 2e^x - x^2 - x - 1$$

is the solution of the considered initial value problem, which is defined for every $x \in \mathbb{R}$.

Now we will apply the last Theorem.

The function

$$f(x,y) = xy + x^2 - x$$

is a continuous function in the rectangle $\overline{S}: |x| \le a, |y-1| \le b$, and

$$|f(x,y)| = |xy + x^2 - x|$$

$$\le |x||y| + |x|^2 + |x|$$

$$\le a(b+1) + a^2 + a$$

$$= a^2 + 2a + ab = M.$$

For all (x, y_1), $(x, y_2) \in \overline{S}$ we have

$$|f(x, y_1) - f(x, y_2)| = |xy_1 + x^2 - x - xy_2 - x^2 + x|$$

$$= |xy_1 - xy_2|$$

$$= |x||y_1 - y_2|$$

$$\leq a|y_1 - y_2|,$$

i.e., the function f satisfies a uniform Lipschitz condition with $L = a$.

Also, $y_0(x) \equiv 1$ in $|x| \leq a$ and

$$|y_0(x) - 1| = 0 \leq b,$$

$$\frac{|\hat{T}'(x)|}{\hat{T}(x)} = \frac{e^x}{e^x} = 1,$$

$$\frac{|\hat{T}(x) - x\hat{T}'(x)|}{\hat{T}(x)} = |1 - x|$$

$$\leq |x| + 1$$

$$\leq a + 1,$$

therefore $P = a + 1$.

Thus, there exists a unique solution of the considered initial value problem in the interval

$$|x| \leq h = \min\Big\{a, \frac{b}{(a+1)(a^2 + ab + 2a + b + 1)}\Big\}.$$

We have $y_0(x) \equiv 1$ and

$$|y(x) - y_0(x)| = |2e^x - x^2 - x - 1 - 1|$$

$$\leq 2e^x + x^2 + |x| + 2$$

$$\leq 2e^h + h^2 + h + 2 = N.$$

Further, the iterative scheme for the considered initial value problem takes the form

$$y_{m+1}(x) = 1 + \int_0^x \Big(ty_m(t) + t(t-1)\Big)dt$$

$$= 1 - \frac{x^2}{2} + \frac{x^3}{3} + \int_0^x ty_m(t)dt,$$

$$y_1(x) = 1 - \tfrac{x^2}{2} + \tfrac{x^3}{3} + \int_0^x t dt$$

$$= 1 + \tfrac{x^3}{3},$$

$$|y(x) - y_1(x)| \leq Ne^{(P+PL)h} \min\{1, (P+PL)h\}, \qquad x \in J_h,$$

$$y_2(x) = 1 - \tfrac{x^2}{2} + \tfrac{x^3}{3} + \int_0^x t\Big(1 + \tfrac{t^3}{3}\Big) dt$$

$$= 1 + \tfrac{x^3}{3} + \tfrac{x^5}{15},$$

$$|y(x) - y_2(x)| \leq Ne^{(P+PL)h} \min\Big\{1, \tfrac{(P+PL)^2h^2}{2}\Big\}, \qquad x \in J_h.$$

Definition 2.1. *If the solution of the initial value problem* (1), (2) *exists in the entire interval* $|x - x_0| \leq a$, *we say that the solution exists globally.*

The next result is called a global existence theorem.

Theorem 2.2. *Let the following conditions be satisfied*

(i) $f(x, y)$ *is continuous in the strip* $T : |x - x_0| \leq a, |y| < \infty$,

(ii) $f(x, y)$ *satisfies a uniform Lipschitz condition in* T,

(iii) $y_0(x)$ *is continuous in* $|x - x_0| \leq a$.

Then the sequence $\{y_m(x)\}_{m=1}^{\infty}$ *generated by Picard iterative scheme exists in the entire interval* $|x - x_0| \leq a$, *and converges to the unique solution* $y(x)$ *of the initial value problem* (1), (2).

Proof. For any continuous function $y_0(x)$ in $|x - x_0| \leq a$, as in the proof of the local existence Theorem, can be established the existence of each $y_m(x)$ in $|x - x_0| \leq a$ satisfying $|y_m(x)| < \infty$. Also, as in the proof of the previous Theorem we have that the sequence $\{y_m(x)(\}_{m=1}^{\infty}$ converges to $y(x)$ in $|x - x_0| \leq a$, replacing h by a throughout the proof and recalling that the function $f(x, y)$ satisfies the Lipschitz condition in the strip T. □

Corollary 2.1. *Let* $f(x, y)$ *be continuous in* $\mathbb{R}^2$ *and satisfies a uniform Lipschitz condition in each strip* $T_a : |x| \leq a, |y| < \infty$, *with the Lipschitz constant* L_a. *Then the initial value problem* (1), (2) *has a unique solution which exists for all* x.

Proof. For any x there exists an $a > 0$ such that $|x - x_0| \leq a$. From here and from $T \subset T_{a+|x_0|}$, it follows that the function $f(x, y)$ satisfies the conditions of the previous Theorem in the strip T. Hence, the result follows for any x. □

We will note that there exist positive constants M_1 and M_2 such that

$$\left|\frac{\hat{T}'(x)}{\hat{T}(x)}\right| \leq M_1, \quad \left|1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right| \leq M_2 \quad \text{for} \quad x \in [x_0 - a, x_0 + a].$$

Theorem 2.3. *(iso-Peano's existence theorem) Let f is defined, continuous and bounded function on the strip $T = \{(x, y) \in \mathbb{R}^2 : |x - x_0| \leq a, |y| < \infty\}$. Then the Cauchy problem (1), (2) has a bounded solution $y(x)$ which is defined on $|x - x_0| \leq a$ and*

$$|y(x)| \leq \left(1 + e^{aM_1}\right)(|y_0| + \sup_{(x,y)\in V} |f(x, y)|M_2 a) \quad \text{for} \quad \forall x \in [x_0 - a, x_0 + a].$$

Remark 2.2. *We can consider our main result as a continuation of the well - known Peano's Theorem.*

If we put

$$g(x, y) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right),$$

then g is unbounded function on the strip T. Therefore we can not apply the classical Peano's Theorem for the Cauchy problem (1), (2), because g has to be bounded on T.

Proof. Since f is a bounded function on T then there exists a positive constant M such that

$$|f(x, y)| \leq M \quad \text{for} \quad (x, y) \in T.$$

We will prove our main result for $x \in [x_0, x_0 + a]$. In the same way one can prove the main result for $x \in [x_0 - a, x_0]$.

For $x \in [x_0, x_0 + a]$ we define the sequence $\{y_m(x)\}_{m=1}^{\infty}$ as follows

$$y_m(x) = y_0 \quad \text{for} \quad x \in \left[x_0, x_0 + \tfrac{a}{m}\right],$$

$$y_m(x) = y_0 + \int_{x_0}^{x - \frac{a}{m}} \left(y_m(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt \quad \text{for}$$

$$x \in \left[x_0 + k\tfrac{a}{m}, x_0 + (k+1)\tfrac{a}{m}\right], \quad k = 1, 2, \ldots, m-1.$$

For this sequence we have

1. Let $m \in \mathbb{N}$ is arbitrary chosen.

   If $x \in \left[x_0, x_0 + \frac{a}{m}\right]$ then

   $$|y_m(x)| = |y_0|.$$

If $x \notin \left[x_0, x_0 + \frac{a}{m}\right]$ and $x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right]$ for some $k = 1, 2, \ldots, m-1$, then

$$|y_m(x)| = \left|y_0 + \int_{x_0}^{x-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt\right|$$

$$\leq |y_0| + \int_{x_0}^{x-\frac{a}{m}} \left(|y_m(t)|\left|\frac{\hat{T}'(t)}{\hat{T}(t)}\right| + |f(t, y_m(t))|\left|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right|\right)dt$$

$$\leq |y_0| + \int_{x_0}^{x-\frac{a}{m}} (M_1|y_m(t)| + MM_2)dt$$

$$= |y_0| + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt + MM_2\left(x - \frac{a}{m} - x_0\right)$$

$$\leq |y_0| + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt + MM_2\left(x_0 + (k+1)\frac{a}{m} - \frac{a}{m} - x_0\right)$$

$$\leq |y_0| + MM_2a + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt,$$

i.e. for $x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right]$ we have

$$|y_m(x)| \leq |y_0| + MM_2a + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt$$

$$\leq |y_0| + MM_2a + M_1 \int_{x_0}^{x} |y_m(t)|dt.$$

From here and the Gronwall's inequality we get

$$|y_m(x)| \leq |y_0| + MM_2a + M_1 \int_{x_0}^{x} (|y_0| + MM_2a)e^{M_1(x-t)}dt$$

$$= |y_0| + MM_2a + e^{M_1x}M_1(|y_0| + MM_2a) \int_{x_0}^{x} e^{-M_1t}dt$$

$$= |y_0| + MM_2a + e^{M_1x}(|y_0| + MM_2a)\left(e^{-M_1x_0} - e^{-M_1x}\right)$$

$$\leq |y_0| + MM_2a + e^{M_1(x-x_0)}(|y_0| + MM_2a)$$

$$\leq |y_0| + MM_2a + e^{aM_1}(|y_0| + MM_2a)$$

$$= (1 + e^{aM_1})(|y_0| + MM_2a) =: M_3 \quad \text{for} \quad x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right],$$

for some $k = 1, 2, \ldots, m-1$.

Consequently for every $x \in [x_0, x_0 + a]$ we have

$$|y_m(x)| \leq M_3 \tag{9}$$

for every $m \in \mathbb{N}$.

Therefore the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is uniformly bounded on $[x_0, x_0+a]$.

2. Let $x_1, x_2 \in [x_0, x_0+a]$ and $m \in \mathbb{N}$ is arbitrarily chosen. Then

1. case. $x_1, x_2 \in \left[x_0, x_0+\frac{a}{m}\right]$. Then

$$y_m(x_1) = y_m(x_2) = y_0,$$

and therefore

$$|y_m(x_2) - y_m(x_1)| = 0.$$

2. case. Let $x_1 \in \left[x_0, x_0+\frac{a}{m}\right]$, $x_2 \notin \left[x_0, x_0+\frac{a}{m}\right]$. Then there exists $k \in \{1, 2, \ldots,$
$m-1\}$, such that $x_2 \in \left[x_0+k\frac{a}{m}, x_0+(k+1)\frac{a}{m}\right]$ and

$$y_m(x_1) = y_0,$$

$$y_m(x_2) = y_0 + \int_{x_0}^{x_2-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt,$$

from here,

$$|y_m(x_2) - y_m(x_1)| = \left|\int_{x_0}^{x_2-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt\right|$$

$$\leq \int_{x_0}^{x_2-\frac{a}{m}} \left(|y_m(t)|\left|\frac{\hat{T}'(t)}{\hat{T}(t)}\right| + |f(t, y_m(t))|\left|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right|\right)dt$$

$$\leq MM_2\left(x_2 - \frac{a}{m} - x_0\right) + M_1 \int_{x_0}^{x_2-\frac{a}{m}} |y_m(t)|dt$$

now we use that $x_1 \in \left[x_0, x_0+\frac{a}{m}\right]$

$$\leq MM_2(x_2 - x_1) + M_1 \int_{x_0}^{x_2-\frac{a}{m}} |y_m(t)|dt,$$

i.e.

$$|y_m(x_2) - y_m(x_1)| \leq MM_2(x_2 - x_1) + M_1 \int_{x_0}^{x_2-\frac{a}{m}} |y_m(t)|dt.$$

From here and (9) we obtain

$$|y_m(x_2) - y_m(x_1)| \leq MM_2(x_2 - x_1) + M_1M_3 \int_{x_0}^{x_2-\frac{a}{m}} dt$$

$$= MM_2(x_2 - x_1) + M_1M_3\left(x_2 - \frac{a}{m} - x_0\right)$$

$$\leq (MM_2 + M_1M_3)(x_2 - x_1).$$

3. case. Let $x_1, x_2 \notin \left[x_0, x_0 + \frac{a}{m}\right]$. Without loss of generality we can suppose that $x_1 \leq x_2$. Let

$$x_1 \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right], \quad x_2 \in \left[x_0 + i\frac{a}{m}, x_0 + (i+1)\frac{a}{m}\right], \quad k \leq i,$$

$k, i \in \{1, 2, \ldots, m-1\}$.

Then

$$y_m(x_2) = y_0 + \int_{x_0}^{x_2 - \frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right) dt,$$

$$y_m(x_1) = y_0 + \int_{x_0}^{x_1 - \frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right) dt,$$

$$y_m(x_2) - y_m(x_1) = \int_{x_1 - \frac{a}{m}}^{x_2 - \frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right) dt,$$

$$|y_m(x_2) - y_m(x_1)| = \left| \int_{x_1 - \frac{a}{m}}^{x_2 - \frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right) dt \right|$$

$$\leq \int_{x_1 - \frac{a}{m}}^{x_2 - \frac{a}{m}} \left(|y_m(t)| \left| \frac{\hat{T}'(t)}{\hat{T}(t)} \right| + |f(t, y_m(t))| \left| 1 - t\frac{\hat{T}'(t)}{\hat{T}(t)} \right| \right) dt$$

$$\leq M_1 \int_{x_1 - \frac{a}{m}}^{x_2 - \frac{a}{m}} |y_m(t)| dt + MM_2 \int_{x_1 - \frac{a}{m}}^{x_2 - \frac{a}{m}} dt$$

$$= M_1 \int_{x_1 - \frac{a}{m}}^{x_2 - \frac{a}{m}} |y_m(t)| dt + MM_2(x_2 - x_1)$$

now we apply (9)

$$\leq M_1 M_3 \int_{x_1 - \frac{a}{m}}^{x_2 - \frac{a}{m}} dt + MM_2(x_2 - x_1)$$

$$= (M_1 M_3 + MM_2)(x_2 - x_1).$$

From 1, 2, and 3 cases follows that for every $x_1, x_2 \in [x_0, x_0 + a]$ we have

$$|y_m(x_2) - y_m(x_1)| \leq (M_1 M_3 + MM_2)|x_2 - x_1| \quad \text{for} \quad \forall m \in \mathbb{N}. \tag{10}$$

Let $\epsilon > 0$ is arbitrary chosen and fixed. Let $\delta = \frac{\epsilon}{MM_2 + M_1 M_3}$. Then if $x_1, x_2 \in [x_0, x_0 + a]$, $|x_1 - x_2| < \delta$, using (10), we get

$$|y_m(x_2) - y_m(x_1)| \leq (M_1 M_3 + MM_2)|x_2 - x_1|$$

$$< (M_1 M_3 + MM_2)\delta = \epsilon.$$

Consequently $\{y_m(x)\}_{m=1}^{\infty}$ is equip-continuous family on $[x_0, x_0 + a]$.

Therefore there exists a subsequence $\{y_{m_p}(x)\}_{p=1}^{\infty}$ of the sequence $\{y_m(x)\}_{m=1}^{\infty}$ which is uniformly convergent to $y(x)$ on $[x_0, x_0+a]$.

For $y_{m_p}(x)$, $x \in [x_0, x_0+a]$, we have

$$y_{m_p}(x) = y_0 + \int_{x_0}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

$$(11) \quad = y_0 + \int_{x_0}^{x} \Big(y_{m_p}(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

$$+ \int_{x}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt.$$

Since f is a continuous and bounded function on T we have

$$(12) \quad \begin{aligned} &\lim_{p\longrightarrow\infty} \int_{x_0}^{x} \Big(y_{m_p}(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt \\ &= \int_{x_0}^{x} \Big(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt. \end{aligned}$$

Also,

$$\begin{aligned} &\Big| \int_{x}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt \Big| \\ &\leq \int_{x-\frac{a}{m_p}}^{x} \Big(|y_{m_p}(t)|\Big|\frac{\hat{T}'(t)}{\hat{T}(t)}\Big| + |f(t, y_{m_p}(t))|\Big|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big|\Big)dt \\ &\leq M_1 \int_{x-\frac{a}{m_p}}^{x} |y_{m_p}(t)|dt + MM_2\frac{a}{m_p} \end{aligned}$$

now we use (9)

$$\leq (M_1M_3 + MM_2)\frac{a}{m_p} \longrightarrow_{p\longrightarrow\infty} 0.$$

From here and (11), (12), when $p \longrightarrow \infty$, we get

$$y(x) = y_0 + \int_{x_0}^{x} \Big(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

for every $x \in [x_0, x_0+a]$. Therefore y is a solution of the Cauchy problem (1), (2) which is defined on $[x_0, x_0+a]$. From (9) follows that $|y(x)| \leq M_3$ for every $x \in [x_0, x_0+a]$. □

Corollary 2.2. *Let $f(x,y)$ be continuous in $\overline{S}$, and hence there exists a $M > 0$ such that $|f(x,y)| \leq M$ for all $(x,y) \in \overline{S}$. Then the initial value problem* (1), (2) *has at least one solution in J_h.*

Proof. The proof is the same as that of the proof of iso-Peano's existence theorem with some obvious changes. □

Definition 2.2. *(ϵ-approximate solution) A function $y(x)$ defined in J is said to be an ϵ-approximate solution of the iso-differential equation* (1) *if*

1. *$y(x)$ is continuous for all $x \in J$,*

2. *for all $x \in J$ the points $(x, y(x)) \in D$,*

3. *$y(x)$ has piecewise continuous derivative in J which may fail to be defined only for a finite number of points, say $x_1, x_2, \ldots, x_k$,*

4. $\left| y'(x) - y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} - f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \right| \leq \epsilon$ *for all $x \in J$, $x \neq x_i$, $i = 1, 2, \ldots, k$.*

The existence of an ϵ-approximate solution is provided in the following theorem.

Theorem 2.4. *Let $f(x,y)$ be continuous in $\overline{S}$ and hence there exists a $M > 0$ such that $|f(x,y)| \leq M$ for every $(x,y) \in \overline{S}$. Then for all $\epsilon > 0$, there exists an ϵ-approximate solution $y(x)$ of the iso-differential equation* (1) *in the interval J_h such that $y(x_0) = y_0$.*

Proof. Because the function $f(x,y)$ is a continuous function in the closed rectangle $\overline{S}$, it is uniformly continuous in this rectangle. Thus, for a given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ so that

$$|f(x,y) - f(x_1,y_1)| \leq \epsilon,$$

$$\left| y\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - y_1\frac{\hat{T}'(x_1)}{\hat{T}(x_1)} - f(x_1,y_1)\frac{\hat{T}(x_1)-x_1\hat{T}'(x_1)}{\hat{T}(x_1)} \right| \leq \epsilon$$

for all (x,y), $(x_1,y_1) \in \overline{S}$ such that

$$|x - x_1| \leq \delta \qquad \text{and} \qquad |y - y_1| \leq \delta.$$

We shall construct an ϵ-approximate solution in the interval $[x_0, x_0 + h]$. A similar process will define it in the interval $[x_0 - h, x_0]$.

For this aim, we divide the interval $[x_0, x_0 + h]$ into m parts

$$x_0 < x_1 < x_2 \ldots < x_m = x_0 + h$$

such that

$$(13) \qquad x_i - x_{i-1} \leq \min\left\{\delta, \frac{\delta}{P(|y_0|+b+M)}\right\}, \qquad i = 1, 2, \ldots, m.$$

Now we define a function $y(x)$ in the interval $[x_0, x_0 + h]$ in the following manner

$$y(x) = y(x_{i-1}) + (x - x_{i-1})\left(y(x_{i-1})\frac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}, y(x_{i-1}))\frac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}\right),$$

(14)

$$x_{i-1} \le x \le x_i, \qquad i = 1, 2, \ldots, m.$$

Obviously, this function $y(x)$ is continuous and has a piecewise continuous derivative

$$y'(x) = y(x_{i-1})\frac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}, y(x_{i-1}))\frac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})},$$

$x_{i-1} < x < x_i$, $i = 1, 2, \ldots, m$, which fails to be defined only at the points x_i, $i = 1, 2, \ldots, m-1$. Since in each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, the function $y(x)$ is a straight line, to prove that $(x, y(x)) \in \overline{S}$ it suffices to show that

$$|y(x_i) - y_0| \le b$$

for all $i = 1, 2, \ldots, m$.

For this reason, in (14) let $i = 1$ and $x = x_1$ to obtain

$$y(x_1) = y_0 + (x - x_1)\left(y_0\frac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\frac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\right),$$

$$|y(x_1) - y_0| = \left|(x - x_1)\left(y_0\frac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\frac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\right)\right|$$

$$\le (x_1 - x_0)\left(|y_0|\frac{|\hat{T}'(x_0)|}{\hat{T}(x_0)} + |f(x_0, y_0)|\frac{|\hat{T}(x_0) - x_0\hat{T}'(x_0)|}{\hat{T}(x_0)}\right)$$

$$\le (x_1 - x_0)(P|y_0| + MP)$$

$$\le hP(M + |y_0|)$$

$$\le hP(b + |y_0| + M)$$

$$\le \frac{b}{P(b+|y_0|+M)}P(b + |y_0| + M)$$

$$= b.$$

Now let the assertion be true for $i = 1, 2, \ldots, k-1 < m-1$, then from (14)

$$y(x_1) - y_0 = (x_1 - x_0)\Big(y_0 \tfrac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\tfrac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\Big),$$

$$y(x_2) - y(x_1) = (x_2 - x_1)\Big(y(x_1)\tfrac{\hat{T}'(x_1)}{\hat{T}(x_1)} + f(x_1, y(x_1))\tfrac{\hat{T}(x_1) - x_1\hat{T}'(x_1)}{\hat{T}(x_1)},$$

$\ldots$

$$y(x_k) - y(x_{k-1}) = (x_k - x_{k-1})\Big(y(x_{k-1})\tfrac{\hat{T}'(x_{k-1})}{\hat{T}(x_{k-1})} + f(x_{k-1}, y(x_{k-1}))\tfrac{\hat{T}(x_{k-1}) - x_{k-1}\hat{T}'(x_{k-1})}{\hat{T}(x_{k-1})}\Big).$$

From here,

$$y(x_k) - y_0 = \textstyle\sum_{l=1}^{k}(x_l - x_{l-1})\Big(y(x_{l-1})\tfrac{\hat{T}'(x_{l-1})}{\hat{T}(x_{l-1})} + f(x_{l-1}, y(x_{l-1}))\tfrac{\hat{T}(x_{l-1}) - x_{l-1}\hat{T}'(x_{l-1})}{\hat{T}(x_{l-1})}\Big),$$

which gives

$$|y(x_k) - y_0| \le \textstyle\sum_{l=1}^{k}(x_l - x_{l-1})\Big(|y(x_{l-1})|\tfrac{|\hat{T}'(x_{l-1})|}{\hat{T}(x_{l-1})} + |f(x_{l-1}, y(x_{l-1}))|\tfrac{|\hat{T}(x_{l-1}) - x_{l-1}\hat{T}'(x_{l-1})|}{\hat{T}(x_{l-1})}\Big)$$

$$\le \textstyle\sum_{l=1}^{k}(x_l - x_{l-1})\Big((b + |y_0|)P + MP\Big)$$

$$= P(M + b + |y_0|)\textstyle\sum_{l=1}^{k}(x_l - x_{l-1})$$

$$= P(M + b + |y_0|)(x_k - x_0)$$

$$\le P(M + b + |y_0|)h$$

$$\le P(M + b + |y_0|)\tfrac{b}{P(M+b+|y_0|)}$$

$$= b.$$

Finally, if $x_{i-1} < x < x_i$, then from (13) and (14)

$$|y(x) - y(x_{i-1})| = (x - x_i)\Big|y(x_{i-1})\tfrac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}), y(x_{i-1}))\tfrac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}\Big|$$

$$\le (x - x_i)\Big(|y(x_{i-1})|\tfrac{|\hat{T}'(x_{i-1})|}{\hat{T}(x_{i-1})} + |f(x_{i-1}), y(x_{i-1}))|\tfrac{|\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})|}{\hat{T}(x_{i-1})}\Big)$$

$$\le (x_i - x_{i-1})\Big((|y_0| + b)P + MP\Big)$$

$$\le \tfrac{\delta}{P(|y_0|+b+M)}P(M + |y_0| + b)$$

$$= \delta.$$

Therefore

$$\left|y'(x) - y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} - f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}\right|$$

$$= \left|y(x_{i-1})\frac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1},y(x_{i-1}))\frac{\hat{T}(x_{i-1})-x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} - y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} - f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}\right|$$

$$\leq \epsilon$$

for all $x \in J_h$, $x \neq x_i$, $i = 1, 2, \ldots, m-1$. This completes the proof that $y(x)$ is an ϵ-approximate solution of the iso-differential equation (1).

This method of constructing an approximate solution is said to be iso-Cauchy-Euler method. □

Theorem 2.5. *(iso-Cauchy-Peano's existence theorem) Let $f(x,y)$ be continuous in $\overline{S}$ and hence there exists a $M > 0$ such that $|f(x,y)| \leq M$ for every $(x,y) \in \overline{S}$. Then the initial value problem* (1), (2) *has at least one solution in J_h.*

Proof. We shall prove the assertion for the interval $[x_0, x_0 + h]$.

Let $\{\epsilon_m\}_{m=1}^{\infty}$ be a monotonically decreasing sequence of positive numbers such that

$$\lim_{m\longrightarrow\infty} \epsilon_m = 0.$$

For each ϵ_m we construct an ϵ_m-approximate solution $y_m(x)$.

As in the proof of the theorem for existence of ϵ-approximate solutions we have

$$|y_m(x)| \leq b + |y_0|$$

for every $m \in \mathbb{N}$ and for every $x \in J_h$. In other words, the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is uniformly bounded in J_h.

Let x, $x^* \in [x_0, x_0 + h]$. Then

$$|y_m(x) - y_m(x^*)| \leq \left|\int_x^{x^*} \left(|y_m(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t,y_m(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_x^{x^*} \Big(P(|y_0| + b) + MP\Big)dt\right|$$

$$\leq P(M + b + |y_0|)|x - x^*|$$

and from this it follows that the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is equip-continuous.

Consequently the sequence $\{y_m(x)\}_{m=1}^{\infty}$ contains a subsequence $\{y_{m_p}(x)\}_{p=1}^{\infty}$ which converges uniformly in $[x_0, x_0 + h]$ to a continuous function $y(x)$. We

define

$$e_m(x) = \begin{cases} y'_m(x) - y_m(x)\frac{\hat{T}'(x)}{\hat{T}(x)} - f(x, y_m(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} \\ \text{at the points where } y'_m(x) \text{ exists} \\ 0 \quad \text{otherwise.} \end{cases}$$

Then

$$(15) \qquad y_m(x) = y_0 + \int_{x_0}^{x} \Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) + e_m(t) \Big) dt$$

and

$$|e_m(x)| \leq \epsilon_m.$$

Since $f(x, y)$ is continuous in $\overline{S}$ and $y_{m_p}(x)$ converges to $y(x)$ uniformly in $[x_0, x_0 + h]$, the function $f(x, y_{m_p}(x))$ converges to $f(x, y(x))$ uniformly in $[x_0, x_0 + h]$. Thus, by replacing m by m_p in (15) and letting $p \longrightarrow \infty$, we become that $y(x)$ is a solution to the integral equation (4). □

Remark 2.3. *We suppose that all conditions of the iso-Cauchy-Peano's existence theorem are satisfied. Further, let the initial value problem* (1), (2) *has a solution* $y(x)$ *in an interval* $J = (\alpha, \beta)$. *We have*

$$|y(x_2) - y(x_1)| \leq P(M + |y_0| + b)|x_2 - x_1|$$

for every x_1, $x_2 \in J$. *Therefore*

$$y(x_2) - y(x_1) \longrightarrow 0$$

as $x_1, x_2 \longrightarrow \alpha^+$. *Thus, by the Cauchy criterion of convergence we have that*

$$\lim_{x \longrightarrow \alpha^+} y(x)$$

exists.

A similar argument holds for

$$\lim_{x \longrightarrow \beta^-} y(x).$$

Theorem 2.6. *Let all conditions of the iso-Cauchy-Peano's existence theorem be satisfied. Let also,* $y(x)$ *be a solution of the initial value problem* (1), (2) *in the interval* $J = (\alpha, \beta)$. *Then* $y(x)$ *can be extended over the interval* $(\alpha, \beta + \gamma]$ $([\alpha - \gamma, \beta))$ *for some* $\gamma > 0$.

Proof. We define the function $y_1(x)$ as follows.

$$y_1(x) = y(x) \qquad \text{for} \qquad x \in (\alpha, \beta),$$

$$y_1(\beta) = y(\beta - 0).$$

We observe that for all $x \in (\alpha, \beta]$ we have

$$y_1(x) = y(\beta - 0) + \int_\beta^x \Big(y_1(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y(x_0) + \int_{x_0}^\beta \Big(y_1(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$+ \int_\beta^x \Big(y_1(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y(x_0) + \int_{x_0}^x \Big(y_1(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

Therefore the left-hand derivative $y_1'(\beta - 0)$ exists and

$$y_1'(\beta - 0) = y_1(\beta) \frac{\hat{T}'(\beta)}{\hat{T}(\beta)} + f(\beta, y_1(\beta)) \frac{\hat{T}(\beta) - \beta\hat{T}'(\beta)}{\hat{T}(\beta)}.$$

Thus, $y_1(x)$ is a continuation of $y(x)$ in the interval $(\alpha, \beta]$.

Let $y_2(x)$ be a solution to the problem

$$y'(x) = y(x) \frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x)) \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

$$y(\beta) = y_1(\beta),$$

existing in the interval $[\beta, \beta + \gamma]$ for some $\gamma > 0$.

We define the function

$$y_3(x) = \begin{cases} y_1(x) & x \in (\alpha, \beta], \\ y_2(x) & x \in [\beta, \beta + \gamma], \end{cases}$$

which is a continuation of $y(x)$ in the interval $(\alpha, \beta + \gamma]$.

Also,

$$y_3(x) = y_0 + \int_{x_0}^x \Big(y_3(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

for every $x \in (\alpha, \beta+\gamma]$, because for all $x \in [\beta, \beta+\gamma]$ we have

$$y_3(x) = y(\beta - 0) + \int_\beta^x \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y_0 + \int_{x_0}^{\beta} \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}x(t)} \Big) dt$$

$$+ \int_\beta^x \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y_0 + \int_{x_0}^{x} \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

□

Theorem 2.7. *(iso-Lipschitz uniqueness theorem) Let $f(x, y)$ be continuous and satisfies a uniform Lipschitz condition in $\overline{S}$ with a Lipschitz constant L. Then the problem* (1), (2) *has at most one solution in $|x - x_0| \le a$.*

Proof. We suppose that the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$, $x \in [x_0 - a, x_0 + a]$. Then

$$y_1(x) = y_0 + \int_{x_0}^{x} \Big(y_1(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt,$$

$$y_2(x) = y_0 + \int_{x_0}^{x} \Big(y_2(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_2(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt,$$

whereupon

$$y_1(x) - y_2(x) = \int_{x_0}^{x} \Big((y_1(t) - y_2(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_1(t)) - f(t, y_2(t)))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt,$$

and

$$|y_1(x) - y_2(x)| \le \Big| \int_{x_0}^{x} \Big(|y_1(t) - y_2(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \Big) dt \Big|$$

$$\le \Big| \int_{x_0}^{x} \Big(P|y_1(t) - y_2(t)| + LP|y_1(t) - y_2(t)| \Big) dt$$

$$= P(1 + L) \Big| \int_{x_0}^{x} |y_1(t) - y_2(t)| dt \Big|.$$

From the last inequality and Gronwall's type inequality we conclude that

$$|y_1(x) - y_2(x)| = 0 \qquad \text{in} \qquad [x_0 - a, x_0 + a].$$

□

Theorem 2.8. *(iso-Peano's uniqueness theorem) Let $f(x,y)$ be continuous in*

$$\overline{S}_+ = \{(x,y) \in \mathbb{R}^2 : x_0 \le x \le x_0 + a, |y - y_0| \le b\}$$

and nonincreasing in y for all $[x_0, x_0 + a]$. Let also,

$$\hat{T}'(x) \le 0, \qquad \hat{T}(x) - x\hat{T}'(x) \ge 0 \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

Then the problem (1), (2) has at most one solution in $[x_0, x_0 + a]$.

Proof. Let the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$ in $[x_0, x_0+a]$ which differ in $[x_0, x_0 + a]$. We assume that

$$y_2(x) > y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon) \subset [x_0, x_0 + a],$$

while $y_1(x) = y_2(x)$ for $x \in [x_0, x_1]$, i.e., x_1 is the greatest lower bound of the set A consisting of those x for which $y_2(x) > y_1(x)$. This greatest lower bound of the set A exists because the set A is bounded below by x_0 at least. Thus for every $x \in (x_1, x_1 + \epsilon)$ we have

$$f(x, y_1(x)) \ge f(x, y_2(x)),$$

$$f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}} \ge f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}},$$

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} \ge y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}},$$

whereupon

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

$$\ge y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

for all $x \in (x_1, x_1 + \epsilon)$, and from here

$$y_1'(x) \ge y_2'(x) \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon).$$

Hence the function

$$z(x) = y_2(x) - y_1(x)$$

is nonincreasing function in $(x_1, x_1 + \epsilon)$.

Because

$$z(x_1) = y_2(x_1) - y_1(x_1) = 0$$

we obtain

$$z(x) \le z(x_1) = 0 \qquad \text{in} \qquad (x_1, x_1 + \epsilon)$$

or

$$y_2(x) \leq y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon).$$

This contradiction proves that

$$y_1(x) = y_2(x) \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

□

Theorem 2.9. *(iso-Peano's uniqueness theorem) Let $f(x, y)$ be continuous in $\overline{S}_+$ and nondecreasing in y for every $x \in [x_0, x_0 + a]$. Let also,*

$$\hat{T}'(x) \leq 0, \qquad \hat{T}(x) - x\hat{T}'(x) \leq 0 \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

Then the problem (1), (2) *has at most one solution in* $[x_0, x_0 + a]$.

Proof. Let the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$ in $[x_0, x_0 + a]$ which differ in $[x_0, x_0 + a]$. Let

$$y_2(x) > y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon) \subset [x_0, x_0 + a],$$

and

$$y_2(x) = y_1(x) \qquad \text{for} \qquad \forall x \in [x_0, x_1].$$

Therefore for every $x \in (x_1, x_1 + \epsilon)$ we have

$$f(x, y_2(x)) \geq f(x, y_1(x)),$$

$$f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}} \geq f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}},$$

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} \geq y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}},$$

whereupon

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

$$\geq y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

for every $x \in (x_1, x_1 + \epsilon)$. Consequently

$$y_1'(x) \geq y_2'(x) \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon)$$

and then the function

$$z(x) = y_2(x) - y_1(x)$$

is nonincreasing function in $(x_1, x_1 + \epsilon)$, therefore

$$y_2(x) - y_1(x) \leq y_2(x_1) - y_1(x_1) = 0 \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon),$$

which is a contradiction. From here we conclude that $y_1(x) = y_2(x)$ for every $x \in [x_0, x_0 + a]$. □

Lemma 2.1. *(iso-Osgood's lemma) Let $w(z)$ be continuous function in $[0,\infty)$, $w(0)=0$, $z+w(z)>0$ in $(0,\infty)$, $z+w(z)$ be increasing function in $[0,\infty)$, and*

(16) $\lim_{\epsilon\longrightarrow 0+}\int_{\epsilon}^{a}\frac{dz}{z+w(z)}=\infty.$

Let $u(x)$ be a nonnegative continuous function in $[0,a]$. Then the inequality

(17) $u(x)\leq P\int_0^x(u(t)+w(u(t)))dt, \qquad 0<x\leq a,$

implies that $u(x)\equiv 0$ in $[0,a]$.

Proof. We define the function

$$v(x)=\max_{0\leq t\leq x}u(t)$$

and assume that $v(x)>0$ for $0<x\leq a$. Then

$$u(t)\leq v(x) \qquad \text{for} \qquad \forall t\in[0,x].$$

Because $u(x)$ is a continuous function in $[0,a]$ then there exists $x_1\in[0,x]$ such that

$$v(x)=u(x_1).$$

Therefore, using that $z+w(z)$ is an increasing function in $[0,\infty)$,

$$v(x)=u(x_1)\leq P\int_0^{x_1}(u(t)+w(u(t)))dt$$

$$\leq P\int_0^{x_1}(v(t)+w(v(t)))dt$$

$$\leq P\int_0^{x}(v(t)+w(v(t)))dt.$$

Let

$$\overline{v}(x)=P\int_0^x(v(t)+w(v(t)))dt.$$

We have

$$\overline{v}(x)\geq 0, \qquad v(x)\leq\overline{v}(x),$$

and

$$\overline{v}'(x)=P(v(x)+w(v(x)))$$

$$\leq P(\overline{v}(x)+w(\overline{v}(x))),$$

and since

$$\overline{v}(x)+w(\overline{v}(x))\geq 0,$$

then

$$\frac{\overline{v}'(x)}{P(\overline{v}(x)+w(\overline{v}(x)))}.$$

Consequently for $0<\delta<a$ we have

$$\int_\delta^a \frac{d\overline{v}(x)}{P(\overline{v}(x)+w(\overline{v}(x)))} \le \int_\delta^a dx,$$

whereupon

$$\lim_{\delta\longrightarrow 0+} \int_\delta^a \frac{d\overline{v}(x)}{P(\overline{v}(x)+w(\overline{v}(x)))} = \lim_{\delta\longrightarrow 0+} \int_{\overline{v}(\delta)}^{\overline{v}(a)} \frac{dy}{P(y+w(y))}$$

$$\le a,$$

which contradicts with (16). Consequently $u(x)\equiv 0$ in $[0,a]$. □

Theorem 2.10. *(iso-Osgood's uniqueness theorem) Let $f(x,y)$ be continuous in $\overline{S}_+$ and for all (x,y_1), $(x,y_2)\in\overline{S}_+$ it satisfies*

$$|f(x,y_1)-f(x,y_2)|\le w(|y_1-y_2|),$$

where $w(z)$ satisfies all conditions of the iso-Osgood's lemma. Then the problem (1), (2) has at most one solution in $[x_0,x_0+a]$.

Proof. Let $y_1(x)$ and $y_2(x)$ are two solutions of the problem (1), (2) in $[x_0,x_0+a]$. Then, if

$$z(x)=|y_1(x)-y_2(x)|, \qquad x\in[x_0,x_0+a],$$

we have

$$z(x)=\left|\int_{x_0}^x\left((y_1(t)-y_2(t))\frac{\hat{T}'(t)}{\hat{T}(t)}+(f(t,y_1(t))-f(t,y_2(t)))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\le \int_{x_0}^x\left(|y_1(t)-y_2(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)}+|f(t,y_1(t))-f(t,y_2(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt$$

$$\le \int_{x_0}(P|y_1(t)-y_2(t)|+Pw(|y_1(t)-y_2(t)|))dt$$

$$= P\int_{x_0}^x(z(t)+w(z(t)))dt.$$

Let

$$u(x)=z(x_0+x).$$

Therefore

$$u(x)\le P\int_{x_0}^{x_0+x}(z(t)+w(z(t)))dt$$

$$= P\int_0^x(z(x_0+t)+w(z(x_0+t)))dt$$

$$= P\int_0^x(u(t)+w(u(t)))dt.$$

Consequently $u(x)$ satisfies the iso-Osgood's lemma, from where $u(x) \equiv 0$ in $[0, a]$, i.e., $y_1(x) = y_2(x)$ in $[x_0, x_0 + a]$. □

Lemma 2.2. *(iso-Nagumo's lemma) Let $u(x)$ be nonnegative continuous function in $[x_0, x_0 + a]$ and $u(x_0) = 0$, and let $u(x)$ be differentiable at $x = x_0$ with $u'(x_0) = 0$. Then*

$$\int_{x_0}^{x} u(t)dt \leq a \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt, \qquad x \in [x_0, x_0 + a],$$

and the inequality

$$u(x) \leq \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt, \qquad x \in [x_0, x_0 + a],$$

implies that $u(x) = 0$ in $[x_0, x_0 + a]$.

Proof. Let

$$g(x) = \int_{x_0}^{x} u(t)dt - a \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt, \qquad x \in [x_0, x_0 + a].$$

Since

$$\lim_{x \longrightarrow x_0} \frac{u(x)}{x - x_0} = u'(x_0) = 0,$$

then the integral

$$\int_{x_0}^{x} \frac{u(t)}{t - x_0} dt$$

exists for $x \in [x_0, x_0 + a]$.

Also,

$$g'(x) = u(x) - a\frac{u(x)}{x - x_0} = u(x)\frac{x - x_0 - a}{x - x_0} \leq 0$$

for every $x \in [x_0, x_0 + a]$. Therefore g is a nonincreasing function in $[x_0, x_0 + a]$, whereupon

$$g(x) \leq g(x_0) \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a],$$

or

$$\int_{x_0}^{x} u(t)dt \leq a \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt$$

for every $x \in [x_0, x_0 + a]$.

Let now

$$v(x) = \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt, \qquad x \in [x_0, x_0 + a].$$

Then

$$u(x) \leq v(x), \qquad x \in [x_0, x_0 + a],$$

and

$$v'(x) = \frac{u(x)}{x-x_0}$$

$$\leq \frac{v(x)}{x-x_0}, \qquad x \in [x_0, x_0 + a].$$

Consequently

$$\frac{d}{dx}\left(\frac{v(x)}{x-x_0}\right) = \frac{v'(x)(x-x_0)-v(x)}{(x-x_0)^2}$$

$$\leq 0$$

or the function

$$l(x) = \frac{v(x)}{x - x_0}$$

is a nonincreasing function in $[x_0, x_0 + a]$ and since $l(x_0) = 0$, we have that

$$v(x) \leq 0 \qquad \text{in} \qquad [x_0, x_0 + a],$$

from where

$$v(x) = 0 \qquad \text{in} \qquad [x_0, x_0 + a].$$

Consequently $u(x) = 0$ in $[x_0, x_0 + a]$. □

Theorem 2.11. *(iso-Nagumo's theorem) Let $P(a + 1) \leq 1$, $f(x, y)$ be continuous in $\overline{S}_+$ and for all (x, y_1), $(x, y_2) \in \overline{S}_+$ it satisfies*

$$|f(x, y_1) - f(x, y_2)| \leq k|x - x_0|^{-1}|y_1 - y_2|, \qquad x \neq x_0. \qquad k \leq 1.$$

Then the problem (1), (2) has at most one solution in $[x_0, x_0 + a]$.

Proof. Let $y_1(x)$ and $y_2(x)$ are two solutions of the problem (1), (2) in $[x_0, x_0 + a]$. Then for $x \in [x_0, x_0 + a]$ we have

$$|y_1(x) - y_2(x)| \leq \int_{x_0}^{x} \left(|y_1(t) - y_2(t)| \frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt$$

$$\leq \int_{x_0}^{x} \Big(P|y_1(t) - y_2(t)| + k(t - y_0)^{-1}|y_1(t) - y_2(t)|P \Big) dt$$

$$\leq P \int_{x_0}^{x} |y_1(t) - y_2(t)| dt + P \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$\leq aP \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt + P \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$= (a + 1)P \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$\leq \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt.$$

Let

$$u(x) = |y_1(x) - y_2(x)|, \qquad x \in [x_0, x_0 + a].$$

Then $u(x_0) = 0$ and from the mean value theorem we have

$$u'(x_0) = \lim_{h \longrightarrow 0} \frac{u(x_0+h)-u(x_0)}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{|y_1(x_0)+hy_1'(x_0+\theta_1 h)+y_2(x_0)-hy_2'(x_0+\theta_2 h)|}{h}$$

$$= (\mathrm{sgn} h) \lim_{h \longrightarrow 0} |y_1'(x_0 + \theta_1 h) - y_2'(x_0 + \theta_2 h)|$$

$$= 0, \qquad 0 < \theta_1, \theta_2 < 1.$$

Then the conditions of iso-Nagumo's lemma are satisfied and $u(x) = 0$, i.e., $y_1(x) = y_2(x)$ in $[x_0, x_0 + a]$. □

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ON PSEUDO SIMPLICIAL LIE ALGEBRAS

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Abstract

In this paper, we define the notion of pseudo simplicial Lie algebras and give a relation between the crossed modules of Lie algebras and pseudo simplicial Lie algebras with Moore complex of length 1.

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Introduction

Simplicial Lie algebras arise in simplicial homotopical algebra, [7][17] and their homotopy theory is of considerable interest. Kassel and Loday [12] (see also [13] and [16]) introduced crossed modules of Lie algebras as omputational algebraic objects equivalent to simplicial Lie algebras with associated Moore complex of length 1.

Crossed modules were introduced by Whitehead in [18] with a view to capturing the relationship between π_1 and π_2 of a space. Homotopy systems (which would now be called free crossed complexes [5] or totally free crossed chain complexes [3],[4]) were introduced, again by Whitehead, to incorporate the action of π_1 on the higher relative homotopy groups of a CW-complex. They consist of a crossed module at the base and a chain complex of modules over π_1 further up.

Inasaridze(c.f. [10],[11]) constructed homotopy groups of pseudosimplicial groups and nonabelian derived functors with values in the category of groups.

In this paper, we define the notion of pseudo simplicial Lie algebras and give a relation between the crossed modules of Lie algebras and pseudo simplicial Lie algebras with Moore complex of length 1.

The above theorem in some sense, are well known. We give details of the proof as analogous proof can be found in the literature [1], [9].

1 Pseudo Simplicial Lie Algebras

Let k be a commutative ring with identity. All Lie algebras will be over a fixed commutative ring k.

A *pseudo simplicial Lie algebra* $\mathbf{L}$ consists of $\{L_n\}$ together with boundary homomorphisms $\partial_i^n : L_n \to L_{n-1}$, $0 \leq i \leq n$, $(n \neq 0)$ and pseudo

degeneracies $s_i^n : L_n \to L_{n+1}$, $0 \leq i \leq n$, satisfying the following pseudosimplicial identities:

$$\begin{array}{lcll} \partial_i^{n-1}\partial_j^n & = & \partial_{j-1}^{n-1}\partial_i^n & \text{for } i < j \\ \partial_i^{n+1}s_j^n & = & s_{j-1}^{n-1}\partial_i^n & \text{for } i < j \\ \partial_j^{n+1}s_j^n & = & 1 = \partial_{j+1}^{n+1}s_j^n & \\ \partial_i^{n+1}s_j^n & = & s_j^{n-1}\partial_{i-1}^n & \text{for } i > j+1 \end{array}$$

To obtain the definition of *simplicial Lie algebra*, we must add the condition that $s_i^{n+1}s_j^n = s_{j+1}^{n+1}s_i^n$ for $i \leq j$ (see [14]).

A topological interpretation is, for example, the F-construction of Milnor [15], which gives the simplicial group of loops of the suspension of a complex. For an arbitrary simplicial set $\mathbf{K}$ with pole ψ, the group of n-simplicies FK_n is the free group on a family of generators σ in one-to-one correspondence with the n-simplicies $\sigma \in K_n$, with the single relation $(s_{n-1}s_{n-2}\ldots s_0(\psi)) = e_n$, while the boundary and degeneracy homomorphisms are induced by the corresponding mappings of the set $\mathbf{K}$.

For any pseudosimplicial Lie algebra $\boldsymbol{L}$, put $NL_n = L_n \cap Ker\ \partial_0^n \cap \ldots \cap Ker\ \partial_{n-1}^n$, $n \geq 0$, and let d_n be the restriction of ∂_n^n to NL_n, $n > 0$. Then $im\ d_n$ is an ideal of L_{n-1}, and $im\ d_{n+1} \subset Ker\ d_n$ for $n > 0$. This determines the Moore complex $\mathbf{NL} = \{NL_n, d_n\}$. Clearly $\mathbf{NL}$ is independent of the pseudodegeneracies, depending only on the boundary homomorphisms.

The n-dimensional homology of the Moore complex $\mathbf{NL}$ is called the *n-dimensional homotopy module* $\pi_n(\mathbf{L})$ of the pseudosimplicial Lie algebra $\mathbf{L}$, $n \geq 0$.

A mapping $\mathbf{f} : \mathbf{L} \longrightarrow \mathbf{L}'$ induces, in a natural fashion, homomorphisms $\pi_n(\mathbf{f}) : \pi_n(\mathbf{L}) \longrightarrow \pi_n(\mathbf{L}')$, $n \geq 0$.

Let $\mathbf{f}$ and $\mathbf{g}$ be two mappings from $\mathbf{L}$ to $\mathbf{L}'$. The following definition is due to Inassaridze [11]. $\mathbf{f}$ is *pseudohomotopic* to $\mathbf{g}$ if there exist Lie

homomorphisms $h_i^n : L_n \longrightarrow L'_{n+1}$, $0 \leq i \leq n$, such that

$$\begin{array}{rcll} \partial_0^{n+1} h_0^n & = & f_n & \partial_{n+1}^{n+1} h_n^n = g_n, \\ \partial_i^{n+1} h_j^n & = & h_{j-1}^{n-1} \partial_i^n & \text{for} \quad i < j, \\ \partial_{j+1}^{n+1} h_{j+1}^n & = & \partial_{j+1}^{n+1} h_j^n, & \\ \partial_i^{n+1} h_j^n & = & h_j^{n-1} \partial_{i-1}^n & \text{for} \quad i > j+1. \end{array}$$

to obtain the definition of homotopy of $\mathbf{f}$ to $\mathbf{g}$, we must add the following conditions:

$$s_i^{n+1} h_j^n = h_{j+1}^{n+1} s_i^n \text{ for } i \leq j, \text{ and } s_i^{n+1} h_j^n = h_j^{n+1} s_{i-1}^n \text{ for } i > j.$$

A mapping $\mathbf{f} : \mathbf{L} \longrightarrow \mathbf{L}'$ of pseudosimplicial Lie algebras is called *simplicial* if it satisfies the condition $f_{n+1} s_i^n = s_i^n f_n$ for $n \geq 0$, $0 \leq i \leq n$. A simplicial map $\mathbf{f} : \mathbf{L} \longrightarrow \mathbf{L}'$ is called a *weak equivalence* if it induces isomorphisms $\pi_n(\mathbf{L}) \cong \pi_n(\mathbf{L}')$ for $n \geq 0$. A simplicial map $\mathbf{f} : \mathbf{L} \longrightarrow \mathbf{L}'$ called a *fibration* if $f_n : L_n \longrightarrow L'_n$ is surjective for $n \geq 0$.

By a k-truncated pseudosimplicial Lie algebra we mean a collection of algebras $\{L_0, \ldots, L_k\}$ and boundary homomorphisms $\partial_i^n : L_n \longrightarrow L_{n-1}$ for $0 \leq i \leq n$, $0 \leq n \leq k$ and pseudodegeneracies $s_i^n : L_n \longrightarrow L_{n+1}$ for $0 \leq i \leq n$, $0 \leq n \leq k$ which satisfy the pseudosimplicial identities. Clearly by forgetting higher dimensions, any pseudosimplicial Lie algebra $\mathbf{L}$ yields a k-truncated pseudosimplicial Lie algebra $tr^k\mathbf{L}$. The functor tr^k admits a right adjoint $\cos k^k$, called the *k-coskeleton functor*, and a left adjoint functor sk^k called the *k-skeleton functor*. We recall from [8] a brief description of these functors.

Suppose $tr^k(\mathbf{L}) = \{L_0, \ldots, L_k\}$ is a pseudosimplicial Lie algebra. A family of Lie homomorphisms

$$(\delta_0, \ldots, \delta_{k+1}) : X_{k+1} \begin{array}{c} \xrightarrow{\delta_{k+1}} \\ \vdots \\ \xrightarrow[\delta_0]{} \end{array} L_k$$

is the *simplicial kernel* of the family of boundary homomorphisms $(\partial_0, \ldots, \partial_k)$ if it has the following universal property: given any family $(\partial_0, \ldots, \partial_{k+1})$ of $k+2$ homomorphisms $\partial_i : Y \longrightarrow L_k$ satisfying the identities $\partial_i \partial_j = \partial_{j-1} \partial_i$ $(0 \leq i < j \leq k+1)$ with the last part of the truncated pseudosimplicial Lie algebra, there exists a unique homomorphism $f : Y \longrightarrow X_{k+1}$ such that $\delta_i f = \partial_i$. Given the simplicial kernel X_{k+1} the family of homomorphisms

$(\alpha_{n+1,j}, \ldots, \alpha_{1j}\alpha_{0j})$ defined by

$$\alpha_{ij} = \begin{cases} s_{j-1} & i < j \\ id & i = j, i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

satisfies the pseudosimplicial identities with the last part of the truncated pseudosimplicial Lie algebra; hence there exists a unique $s_j : L_k \longrightarrow X_{k+1}$ such that $\delta_i s_j = \alpha_{ij}$. We thus have a $(k+1)$-truncated pseudosimplicial Lie algebra $\{L_0, \ldots, L_k, X_{k+1}\}$. By iterating this construction we get a pseudosimplicial Lie algebra $\cos k^k(tr^k(\mathbf{L})) = \{L_0, \ldots, L_k, X_{k+1}\}$ called the *coskeleton* of the truncated pseudosimplicial Lie algebra. If $\mathbf{L}$, $\mathbf{L}'$ are any pseudosimplicial Lie algebras, than any truncated simplicial map $\mathbf{f} : tr^k\mathbf{L} \longrightarrow tr^k\mathbf{L}'$ extends uniquely to a simplicial map $\mathbf{f} : \mathbf{L} \longrightarrow \cos k^k(tr^k(\mathbf{L}'))$.

The k-skeleton functor can be constructed by a dual process involving pseudosimplicial cokernels

$$(s_0, \ldots, s_k) : L_k \underset{\longrightarrow}{\overset{s_k}{\vdots}} X_{k+1}.$$
$$\qquad\qquad\qquad\qquad\quad\ \ s_0$$

(That is, universal systems of $k+1$ arrows which satisfy pseudosimplicial identities.)

2 Crossed Modules of Lie Algebras

J.H.C. Whitehead (1949) [18] described crossed modules in various contexts especially in his investigation into the group structure of relative homotopy groups. The notion of crossed module of Lie algebras was defined by Kassel and Loday [12].

Let M and P be two Lie algebras. By an action of P on M we mean a bilinear map $P \times M \longrightarrow M$, $(p, m) \longmapsto p \cdot m$ satisfying

$$[p, p'] \cdot m = p \cdot (p' \cdot m) - p' \cdot (p \cdot m)$$

$$p \cdot [m, m'] = [p \cdot m, m'] + [m, p \cdot m']$$

for all $m, m' \in M$, $p, p' \in P$. For instance, if P is a subalgebra of some Lie algebra Q (including possibly the case $P = Q$),and if M is an ideal in Q, then Lie bracket in Q yields an action of P on M.

A crossed module of Lie algebras is a Lie homomorphism $\partial : M \longrightarrow P$ together with an action of P on M such that for all $m, m' \in M$, $p \in P$

$$\textbf{CM1)}\ \partial(p \cdot m) = [p, \partial(m)]$$

and

$$\textbf{CM2)}\ \partial m \cdot m' = [m, m'] \ .$$

The last condition is called the Peiffer identity. We denote such a crossed module by $\mathcal{P} = (M, P, \partial)$.

A morphism of crossed modules Lie algebras from $\mathcal{P} = (M, P, \partial)$ to $\mathcal{P}' = (M', P', \partial')$ is a pair of Lie algebra morphisms,

$$\theta : M \longrightarrow M', \psi : P \longrightarrow P'$$

such that

$$\theta(p \cdot m) = \psi(p) \cdot \theta(m) \text{ and } \partial'\theta(m) = \psi\partial(m).$$

Therefore we can define the category of Lie crossed modules denoting it as **LXmod.**

Examples of Crossed Modules

1. Let I be any ideal of a Lie algebra P. Consider an inclusion map

$$inc : I \longrightarrow P.$$

Then (I, P, inc) is a crossed module. Conversely given any crossed module $\partial : M \longrightarrow P$, one can easily verify that $\partial M = I$ is an ideal in P.

2. Let M be any R-bimodule. It can be consider as an R-algebra with zero multiplication, and then $\mathbf{0} : M \longrightarrow R$ is a crossed R-module by $\mathbf{0}(c) \cdot c' = 0c' = 0 = cc'$ for all $c, c' \in M$.

Conversely, given any crossed module $\partial : C \longrightarrow R$, then $Ker\partial$ is an $R/\partial C$-module.

The following lemma is a straightforward modification of Theorem 1.3 in [6].

Lemma 2.1 *Let* $\mathbf{L}$ *be a pseudosimplicial Lie algebra. The Moore complex of its* k *-coskeleton* $\cos k^k(tr^k\mathbf{L})$ *is of length* $k+1$*, and is identical to the Moore complex of* $\mathbf{L}$ *in dimensions* $\leq k$*. Moreover, in dimensions* $k-1$ *to* $k+2$ *the Moore complex of* $\cos k^k(tr^k\mathbf{L})$ *is an exact sequence*

$$0 \longrightarrow N(\cos k^k(tr^k\mathbf{L}))_{k+1} \xrightarrow{\partial_{k+1}} NL_k \xrightarrow{\partial_k} NL_{k-1}.$$

where N_k *is the kth term of the Moore complex of* $\mathbf{L}$.

Proof: The $(k+1)-$dimensional part of $\cos k^k\left(tr^k\mathbf{L}\right)$ can be identified with the subalgebra of the $(k+2)-$ fold direct sum L_k^{k+2} consisting of those elements $(x_0, ..., x_{k+1})$ such that $d_j x_k = d_{k-1}x_j$ for $j < k$; the face maps are given by $d_j\,(x_0, ..., x_{k+1}) = x_j$ Thus $N(\cos k^k(tr^k\mathbf{L}))_{k+1}$consists of elements $(0, ..., 0, x_{k+1})$ such that $d_j x_{k+1} = 0$ for all j. In other words

$N(\cos k^k(tr^k\mathbf{L}))_{k+1}$ is the kernel of $\partial_k : NL_k \longrightarrow NL_{k-1}$, and hence we have the exact sequence of the lemma.

The injectivity of ∂_{k+1} and the isomorphism

$$\cos k^{n-1}\left(tr^{n-1}\left(\cos k^k\left(tr^kL\right)\right)\right) \simeq \cos k^k\left(tr^kL\right)$$

for $n \geq k+2$ shows that the Moore complex of $\cos k^k\left(tr^kL\right)$ is of length $k+1$. □

The following theorem is well known. In [9] this theorem was proved.

Theorem 2.2 *([9]) The category of crossed modules of Lie algebras is equivalent to the category of simplicial Lie algebras with Moore complex of length 1.*

Now, we shall give the pseudo version of this theorem.

Theorem 2.3 *The category of crossed modules of Lie algebras is equivalent to the category of pseudosimplicial Lie algebras with Moore complex of length 1.*

Proof: Let $\mathbf{L}$ be a pseudosimplicial Lie algebra with Moore complex of length 1. Put $P = NL_0 = L_0$, $M = NL_1 = \ker(d_0 : L_1 \longrightarrow L_0)$ and $\partial = d_1$ (restricted to M). Then $p \in P$ acts on $m \in M$ by $p \cdot m = [s_0(p), m]$, and $\partial(p \cdot m) = d_1[s_0(p), m]$. Since the Moore complex $\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\partial} P \longrightarrow 0$ is of length 1 we have $\partial_2 NE_2 = 0$. It then follows that for all $m, m' \in M$ and $p \in P$,

$$\begin{aligned}(i)\quad \partial_1(p \cdot m) &= d_1(p \cdot m)\\ &= d_1[s_0(p), m]\\ &= [d_1 s_0(p), d_1(m)]\\ &= [p, \partial_1(m)]\end{aligned}$$

$$
\begin{aligned}
(ii)\quad (\partial_1 m)\cdot m' &= [s_0(\partial_1(m)), m'] \\
&= [s_0(d_1(m)), m'] \\
&= [s_0(d_1(m)), m'] - [m, m'] + [m, m'] \\
&= [(s_0(d_1(m)) - m), m'] + [m, m'] \\
&= [(d_2 s_0(m) - d_2 s_1(m)), d_2 s_1(m')] + [m, m'] \\
&= d_2[(s_0(m) - s_1(m)), s_1(m')] + [m, m'] \\
&= [m, m']
\end{aligned}
$$

for $m, m' \in M$ because $(s_0(m) - s_1(m))s_1(m')$ lies in NL_2. Thus $\partial : M \longrightarrow P$ is a crossed module of Lie algebras.

Conversely, let $\partial : M \longrightarrow P$ be a crossed module of Lie algebras. By using the action of P on M we can form the semi-direct product $M \rtimes P = \{(m,p) : m \in M\ ,\ p \in P\}$, in which scalar multiplication is $\lambda(m,p) = (\lambda m, \lambda p)$, addition is $(m,p) + (m',p') = (m+m', p+p')$ and multiplication is

$$[(m,p),(m',p')] = ([m,m'] + p\cdot m' - p'\cdot m, [p,p'])$$

for $m, m' \in M$, $p, p' \in P$. There are Lie homomorphisms

$$
\begin{aligned}
d_o &: M \rtimes P \longrightarrow P, \qquad (m,p) \longmapsto p, \\
d_1 &: M \rtimes P \longrightarrow P, \qquad (m,p) \longmapsto (\partial m) + p, \\
s_0 &: P \longrightarrow M \rtimes P, \qquad p \longmapsto (0,p)\,.
\end{aligned}
$$

Let $L_0 = P$, $L_1 = M \rtimes P$. We have a 1 - truncated pseudosimplicial Lie algebra $\{L_0, L_1\}$ whose 1- coskeleton we denote by L^1. The algebra $M \rtimes P$ acts on M via the action of P on M and the Lie homomorphism d_1. We can thus form the semi-direct product $M \rtimes (M \rtimes P)$ and construct homomorphisms

$$
\begin{aligned}
d_0 &: M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, \qquad (m, m', p) \longmapsto (m', p)\,, \\
d_1 &: M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, \qquad (m, m', p) \longmapsto (m + m', p)\,, \\
d_2 &: M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, \qquad (m, m', p) \longmapsto (m, (\partial m') + p)\,, \\
s_0 &: M \rtimes P \longrightarrow M \rtimes (M \rtimes P)\,, \qquad (m,p) \longmapsto (0, m, p)\,, \\
s_1 &: M \rtimes P \longrightarrow M \rtimes (M \rtimes P)\,, \qquad (m,p) \longmapsto (m, 0, p)\,.
\end{aligned}
$$

Conditions (i) and (ii) of a crossed module ensure that these are homomorphisms (Condition (ii) is needed for d_2). Let $L_2 = M \rtimes (M \rtimes P)$. We then have a 2 - truncated pseudosimplicial Lie algebra $\{L_0, L_1, L_2\}$ whose 2 - coskeleton we denote by L^2. There is a unique simplicial map $L^2 \longrightarrow L^1$ which in dimensions 0 and 1 is the identity. We let $\overline{L}^2$ denote the image of L^2 in L^1. It is readily checked that the Moore comple of L^2 is trivial in dimension 2; it follows from Lemma 2.3 that $\overline{L}^2$ is a pseudosimplicial Lie algebra whose Moore complex is of length 1. □

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JORDAN n-TUPLE MAPS ON TRIANGULAR ALGEBRAS

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Abstract

Let $\mathfrak{T}$ be a triangular algebra and $\mathfrak{S}$ an arbitrary ring. We study the additivity of surjective maps $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$ preserving the n-tuple Jordan product on these rings. In this paper, we give sufficient conditions on $\mathfrak{T}$ such that both M and M^* are additives. In particular, if $\mathfrak{T}$ is a standard subalgebra of a nest algebra, then M and M^* are additive.

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1 Triangular algebras and Jordan n-tuple maps of $\mathfrak{T} \times \mathfrak{S}$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras over a commutative ring $\mathfrak{R}$ and $\mathfrak{X}$ a faithful $(\mathfrak{A}, \mathfrak{B})$-bimodule (see [3]). The $\mathfrak{T}$-algebra

$$\mathrm{Tri}(\mathfrak{A}, \mathfrak{X}, \mathfrak{B}) = \left\{ \begin{pmatrix} a & x \\ & b \end{pmatrix} \ : \ a \in \mathfrak{A},\ x \in \mathfrak{X},\ b \in \mathfrak{B} \right\}$$

under the usual matrix addition and formal matrix multiplication will be called a *triangular algebra*. An important example of triangular algebras are the nest algebras (see [1]). Let $\mathfrak{T} = \mathrm{Tri}(\mathfrak{A}, \mathfrak{X}, \mathfrak{B})$ be a triangular algebra. We set

$$\mathfrak{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} \ : \ a \in \mathfrak{A} \right\},$$

$$\mathfrak{T}_{12} = \left\{ \begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} \ : \ x \in \mathfrak{X} \right\},$$

and

$$\mathfrak{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} \ : \ b \in \mathfrak{B} \right\}.$$

Then we can write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$ and every element $c \in \mathfrak{T}$ can be written as $c = c_{11} + c_{12} + c_{22}$, where $c_{ij} \in \mathfrak{T}_{ij}$ $(1 \leq i \leq j \leq 2)$. Note that $a_{ij} b_{kl} = 0$ if $j \neq k$.

Let $\mathfrak{T} = \mathrm{Tri}(\mathfrak{A}, \mathfrak{X}, \mathfrak{B})$ be a triangular algebra, $\mathfrak{S}$ an arbitrary ring and $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$ two maps. We call the ordered pair (M, M^*) a *Jordan n-tuple map of* $\mathfrak{T} \times \mathfrak{S}$ if

$$M(M^*(x_n) \circ (\cdots (M^*(x_2) \circ a_1) \cdots)) = x_n \circ (\cdots (x_2 \circ M(a_1)) \cdots), \quad (1)$$

$$M^*(M(a_n) \circ (\cdots (M(a_2) \circ x_1) \cdots)) = a_n \circ (\cdots (a_2 \circ M^*(x_1)) \cdots), \quad (2)$$

for all $a_1, \cdots, a_n \in \mathfrak{T}$ and $x_1, \cdots, x_n \in \mathfrak{S}$.

We say that a Jordan n-tuple map (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$ is *additive* (resp., *injective, surjective, bijective*) if both maps M and M^* are additive (resp., injective, surjective, bijective).

It is an interesting problem to study the interrelation between the multiplicative and the additive structure of a ring. In particular, Ling and Lu [2] showed that every Jordan 2-tuple map on a standard subalgebra of a nest algebra is additive. Later, Peisheng [3] extended this result to some triangular algebras. In this paper, we generalize the result of Peisheng to the class of Jordan n-tuple maps. As a consequence, we also obtained an application on a class of standard subalgebras of a nest algebra.

2 The main result

Our main result reads as follows.

Theorem 2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras over a commutative ring $\mathfrak{R}$, $\mathfrak{X}$ a faithful $(\mathfrak{A},\mathfrak{B})$-bimodule and let $\mathfrak{T}$ be the triangular algebra* $\mathrm{Tri}(\mathfrak{A},\mathfrak{X},\mathfrak{B})$ *satisfying the following conditions:*

(i) If $a_1 \circ (\cdots (a_1 \circ a)\cdots) = 0$ for all $a_1 \in \mathfrak{A}$, then $a = 0$;

(ii) If $b_1 \circ (\cdots (b_1 \circ b)\cdots) = 0$ for all $b_1 \in \mathfrak{B}$, then $b = 0$;

(iii) If $x \in \mathfrak{X}$ is such that $ax = 0$, for all $a \in \mathfrak{A}$, or $xb = 0$, for all $b \in \mathfrak{B}$, then $x = 0$.

Let $\mathfrak{S}$ be an arbitrary ring. Then every surjective Jordan n-tuple map (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$ is additive.

Based on the techniques used by Ling and Lu [2] and Peisheng [3], we shall organize the proof of Theorem 2.1 in a serie of lemmas. We observe that many formulas and relations presented in this paper are based in the mathematical induction principle. For sake of clarity of the whole text we have omitted their proofs. We begin with the following lemma:

Lemma 2.1. *$M(0) = 0$ and $M^*(0) = 0$.*

Proof. From the identity (1) we have

$$\begin{aligned} M(0) &= M(M^*(0) \circ (\cdots (M^*(0) \circ 0)\cdots)) = 0 \circ (\cdots (0 \circ M(0))\cdots) \\ &= 0. \end{aligned}$$

Similarly, we prove that $M^*(0) = 0$, from the identity (2). □

Lemma 2.2. *M and M^* are bijective.*

Proof. From the hypothesis of the Theorem 2.1, it suffices to prove that (M, M^*) is injective. First we show that M is injective. Let a and b be in $\mathfrak{T}$ and suppose that $M(a) = M(b)$. From the identity (2) we have

$$\begin{aligned} a_n \circ (\cdots (a_3 \circ (a \circ M^*(x_1))) \cdots) \\ &= M^*(M(a_n) \circ (\cdots (M(a_3) \circ (M(a) \circ x_1)) \cdots)) \\ &= M^*(M(a_n) \circ (\cdots (M(a_3) \circ (M(b) \circ x_1)) \cdots)) \\ &= a_n \circ (\cdots (a_3 \circ (b \circ M^*(x_1))) \cdots) \end{aligned}$$

for all $a_3, \cdots, a_n \in \mathfrak{T}$ and $x_1 \in \mathfrak{S}$, which implies

$$a_n \circ (\cdots (a_3 \circ ((a - b) \circ M^*(x_1))) \cdots) = 0. \tag{3}$$

From the surjectivity of M^* for all $a^k_{i_k j_k} \in \mathfrak{T}_{i_k j_k}$ $(1 \leq i_k \leq j_k \leq 2; k = 1, 3, \cdots, n)$, there is $x_1 = x_1(a^1_{i_1 j_1}) \in \mathfrak{S}$ such that $a^1_{i_1 j_1} = M^*(x_1(a^1_{i_1 j_1}))$ and

$$a^n_{i_n j_n} \circ (\cdots (a^3_{i_3 j_3} \circ ((a - b) \circ a^1_{i_1 j_1})) \cdots) = 0 \tag{4}$$

by (3). This shows that, if $(i_1, j_1) = (1, 1)$ and $(i_k, j_k) = (2, 2)$ $(k = 3, \cdots, n)$, then from (4) we obtain $a^1_{i_1 j_1}(a_{12} - b_{12}) a^3_{i_3 j_3} \cdots a^n_{i_n j_n} = 0$. Thus $a_{12} - b_{12} = 0$, by condition (iii) of the Theorem. Also, if $(i_k, j_k) = (1, 1)$ $(k = 1, 3, \cdots, n)$, then from (4) we obtain $a^n_{i_n j_n} \circ (\cdots (a^3_{i_3 j_3} \circ ((a_{11} - b_{11}) \circ a^1_{i_1 j_1})) \cdots) = 0$. Thus $a_{11} - b_{11} = 0$, by condition (i) of the Theorem. Yet, if $(i_k, j_k) = (2, 2)$ $(k = 1, 3 \cdots, n)$, then from (4) again we have $a_{22} - b_{22} = 0$, by condition (ii) of the Theorem. This allows us to conclude that $a = b$.

Now we prove the injectivity of M^*. Let x and y be in $\mathfrak{S}$ and suppose $M^*(x) = M^*(y)$. From the identities (1) and (2) and that M is a bijective map, we have

$$\begin{aligned} & M^{-1}(x_n) \circ (\cdots (M^{-1}(x_3) \circ (M^{-1}(x) \circ M^* M(a_1))) \cdots) \\ = \; & M^*(MM^{-1}(x_n) \circ (\cdots (MM^{-1}(x_3) \circ (MM^{-1}(x) \circ M(a_1))) \cdots)) \\ = \; & M^*(x_n \circ (\cdots (x_3 \circ (x \circ M(a_1))) \cdots)) \\ = \; & M^* M(M^*(x_n) \circ (\cdots (M^*(x_3) \circ (M^*(x) \circ a_1)) \cdots)) \\ = \; & M^* M(M^*(x_n) \circ (\cdots (M^*(x_3) \circ (M^*(y) \circ a_1)) \cdots)) \end{aligned}$$

$$
\begin{aligned}
&= M^*(x_n \circ (\cdots (x_3 \circ (y \circ M(a_1))) \cdots)) \\
&= M^*(MM^{-1}(x_n) \circ (\cdots (MM^{-1}(x_3) \circ (MM^{-1}(y) \circ M(a_1))) \cdots)) \\
&= M^{-1}(x_n) \circ (\cdots (M^{-1}(x_3) \circ (M^{-1}(y) \circ M^*M(a_1))) \cdots)
\end{aligned}
$$

for all $a_1 \in \mathfrak{T}$ and $x_3, \cdots, x_n \in \mathfrak{S}$, which implies

$$M^{-1}(x_n) \circ (\cdots (M^{-1}(x_3) \circ ((M^{-1}(x) - M^{-1}(y)) \circ M^*M(a_1))) \cdots) = 0.$$

Since M^*M is also a surjective map and using a similar argument developed in the identity (4), we see that $M^{-1}(x) = M^{-1}(y)$. Therefore $x = y$. □

Lemma 2.3. *The pair (M^{*-1}, M^{-1}) is a Jordan n-tuple map of $\mathfrak{T} \times \mathfrak{S}$.*

Proof. First, we observe that by identity (2)

$$
\begin{aligned}
& M^*(x_n \circ (\cdots (x_2 \circ M^{*-1}(a_1)) \cdots)) \\
= \; & M^*(MM^{-1}(x_n) \circ (\cdots (MM^{-1}(x_2) \circ M^{*-1}(a_1)) \cdots)) \\
= \; & M^{-1}(x_n) \circ (\cdots (M^{-1}(x_2) \circ a_1) \cdots),
\end{aligned}
$$

for all $a_1 \in \mathfrak{T}$ and $x_2, \cdots, x_n \in \mathfrak{S}$. Thus,

$$M^{*-1}(M^{-1}(x_n) \circ (\cdots (M^{-1}(x_2) \circ a_1) \cdots)) = x_n \circ (\cdots (x_2 \circ M^{*-1}(a_1)) \cdots),$$

by Lemma 2.2.

The other identity is proved similarly. □

Lemma 2.4. *Let $a, b, c \in \mathfrak{T}$ such that $M(c) = M(a) + M(b)$. Then*

$$
\begin{aligned}
M(a_n \circ (\cdots (a_2 \circ c) \cdots)) = {} & M(a_n \circ (\cdots (a_2 \circ a) \cdots)) \\
& + M(a_n \circ (\cdots (a_2 \circ b) \cdots))
\end{aligned}
$$

for all $a_2, \cdots, a_n \in \mathfrak{T}$.

Proof. For arbitrary elements $a_2, \cdots, a_n \in \mathfrak{T}$ we have, by identity (1),

$$
\begin{aligned}
& M(a_n \circ (\cdots (a_2 \circ c) \cdots)) \\
= \; & M(M^*M^{*-1}(a_n) \circ (\cdots (M^*M^{*-1}(a_2) \circ c) \cdots))
\end{aligned}
$$

$$
\begin{aligned}
&= M^{*-1}(a_n) \circ (\cdots (M^{*-1}(a_2) \circ M(c)) \cdots) \\
&= M^{*-1}(a_n) \circ (\cdots (M^{*-1}(a_2) \circ (M(a) + M(b))) \cdots) \\
&= M^{*-1}(a_n) \circ (\cdots (M^{*-1}(a_2) \circ M(a)) \cdots) \\
&\quad + M^{*-1}(a_n) \circ (\cdots (M^{*-1}(a_2) \circ M(b)) \cdots) \\
&= M(M^* M^{*-1}(a_n) \circ (\cdots (M^* M^{*-1}(a_2) \circ a) \cdots)) \\
&\quad + M(M^* M^{*-1}(a_n) \circ (\cdots (M^* M^{*-1}(a_2) \circ b) \cdots)) \\
&= M(a_n \circ (\cdots (a_2 \circ a) \cdots)) + M(a_n \circ (\cdots (a_2 \circ b) \cdots)).
\end{aligned}
$$

□

From the Lemma 2.3, we can obtain the following similar result.

Lemma 2.5. *Let $a, b, c \in \mathfrak{T}$ such that $M^{*-1}(c) = M^{*-1}(a) + M^{*-1}(b)$. Then*

$$
\begin{aligned}
M^{*-1}(a_n \circ (\cdots (a_2 \circ c) \cdots)) = M^{*-1}&(a_n \circ (\cdots (a_2 \circ a) \cdots)) \\
&+ M^{*-1}(a_n \circ (\cdots (a_2 \circ b) \cdots))
\end{aligned}
$$

for all $a_2, \cdots, a_n \in \mathfrak{T}$.

Lemma 2.6. *For arbitrary elements $a_{11} \in \mathfrak{T}_{11}$ and $b_{12} \in \mathfrak{T}_{12}$ the following hold:*

(i) $M(a_{11} + b_{12}) = M(a_{11}) + M(b_{12})$;

(ii) $M^{*-1}(a_{11} + b_{12}) = M^{*-1}(a_{11}) + M^{*-1}(b_{12})$.

Proof. From the bijectivity of M, there is a single element $c = c_{11} + c_{12} + c_{22} \in \mathfrak{T}$ such that $M(c) = M(a_{11}) + M(b_{12})$. Hence, for arbitrary elements $a^k_{i_k j_k} \in \mathfrak{T}_{i_k j_k}$ $(1 \le i_k \le j_k \le 2; k = 2, \cdots, n)$, with $(i_2, j_2) = (1, 1)$, $(i_k, j_k) = (2, 2)$ $(k = 3, \cdots, n)$ and the Lemma 2.4 we have

$$
\begin{aligned}
&M(a^n_{22} \circ (\cdots (a^3_{22} \circ (a^2_{11} \circ c)) \cdots)) \\
= \; &M(a^n_{22} \circ (\cdots (a^3_{22} \circ (a^2_{11} \circ a_{11})) \cdots)) \\
&+ M(a^n_{22} \circ (\cdots (a^3_{22} \circ (a^2_{11} \circ b_{12})) \cdots)) \\
= \; &M(a^n_{22} \circ (\cdots (a^3_{22} \circ (a^2_{11} \circ b_{12})) \cdots)).
\end{aligned}
$$

It follows that $a_{22}^n \circ (\cdots (a_{22}^3 \circ (a_{11}^2 \circ c)) \cdots) = a_{22}^n \circ (\cdots (a_{22}^3 \circ (a_{11}^2 \circ b_{12})) \cdots)$ which implies $a_{11}^2(c_{12} - b_{12})a_{22}^3 \cdots a_{22}^n = 0$. Thus $c_{12} = b_{12}$, by condition (iii) of the Theorem. Next, for arbitrary elements $a_{i_k j_k}^k \in \mathfrak{T}_{i_k j_k}$ $(1 \leq i_k \leq j_k \leq 2; k = 2, \cdots, n)$, with $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n)$ and the Lemma 2.4 we have

$$\begin{aligned} & M(a_{22}^n \circ (\cdots (a_{22}^2 \circ c) \cdots)) \\ = \; & M(a_{22}^n \circ (\cdots (a_{22}^2 \circ a_{11}) \cdots)) \\ & + M(a_{22}^n \circ (\cdots (a_{22}^2 \circ b_{12}) \cdots)) \\ = \; & M(a_{22}^n \circ (\cdots (a_{22}^2 \circ b_{12}) \cdots)). \end{aligned}$$

This results that $a_{22}^n \circ (\cdots (a_{22}^2 \circ c) \cdots) = a_{22}^n \circ (\cdots (a_{22}^2 \circ b_{12}) \cdots)$ which implies $a_{22}^n \circ (\cdots (a_{22}^n \circ c_{22}) \cdots) = 0$. Therefore $c_{22} = 0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $a_{i_k j_k}^k \in \mathfrak{T}_{i_k j_k}$ $(1 \leq i_k \leq j_k \leq 2; k = 2, \cdots, n)$, with $(i_2, j_2) = (1,2)$ and $(i_k, j_k) = (1,1)$ $(k = 3, \cdots, n)$ we have

$$\begin{aligned} & M(a_{11}^n \circ (\cdots (a_{11}^3 \circ (a_{12}^2 \circ c)) \cdots)) \\ = \; & M(a_{11}^n \circ (\cdots (a_{11}^3 \circ (a_{12}^2 \circ a_{11})) \cdots)) \\ & + M(a_{11}^n \circ (\cdots (a_{11}^3 \circ (a_{12}^2 \circ b_{12})) \cdots)) \\ = \; & M(a_{11}^n \circ (\cdots (a_{11}^3 \circ (a_{12}^2 \circ a_{11})) \cdots)). \end{aligned}$$

This shows that $a_{11}^n \circ (\cdots (a_{11}^3 \circ (a_{12}^2 \circ c)) \cdots) = a_{11}^n \circ (\cdots (a_{11}^3 \circ (a_{12}^2 \circ a_{11})) \cdots)$ which implies $a_{11}^n \cdots a_{11}^3 (c_{11} - a_{11}) a_{12}^2 = 0$. Therefore $c_{11} = a_{11}$, by condition (iii) of the Theorem and the fact that $\mathfrak{T}$ is a faithfull bimodule.

Using the same argument as in the previous case we can prove (ii), by Lemma 2.5. □

Similarly, we prove the following lemma.

Lemma 2.7. *For arbitrary elements $a_{12} \in \mathfrak{T}_{12}$ and $b_{22} \in \mathfrak{T}_{22}$ the following hold:*

(i) $M(a_{12} + b_{22}) = M(a_{12}) + M(b_{22})$;

(ii) $M^{*-1}(a_{12} + b_{22}) = M^{*-1}(a_{12}) + M^{*-1}(b_{22})$.

Lemma 2.8. *For arbitrary elements $a_{12}, b_{12} \in \mathfrak{T}_{12}$, $c_{11}^1, \cdots, c_{11}^{n-1} \in \mathfrak{T}_{11}$ and $d_{22}^1, \cdots, d_{22}^{n-1} \in \mathfrak{T}_{22}$ we have:*

(i) $M(c_{11}^{n-1}\cdots c_{11}^{1}a_{12}+c_{11}^{n-1}\cdots c_{11}^{1}b_{12})$
$$= M(c_{11}^{n-1}\cdots c_{11}^{1}a_{12})+M(c_{11}^{n-1}\cdots c_{11}^{1}b_{12});$$

(ii) $M(c_{11}^{n-1}\cdots c_{11}^{2}a_{12}d_{22}^{1}+c_{11}^{n-1}\cdots c_{11}^{2}b_{12}d_{22}^{1})$
$$= M(c_{11}^{n-1}\cdots c_{11}^{2}a_{12}d_{22}^{1})+M(c_{11}^{n-1}\cdots c_{11}^{2}b_{12}d_{22}^{1});$$

(iii) $M(a_{12}d_{22}^{1}\cdots d_{22}^{n-1}+b_{12}d_{22}^{1}\cdots d_{22}^{n-1})$
$$= M(a_{12}d_{22}^{1}\cdots d_{22}^{n-1})+M(b_{12}d_{22}^{1}\cdots d_{22}^{n-1}).$$

Proof. (i) First, we note that the following identity is valid

$$\begin{aligned} & c_{11}^{n-1}\cdots c_{11}^{1}a_{12}+c_{11}^{n-1}\cdots c_{11}^{1}b_{12}+c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ(c_{11}^{1}\circ c_{11}^{1}))\cdots) \\ = \; & c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ((a_{12}+c_{11}^{1})\circ(b_{12}+c_{11}^{1})))\cdots). \end{aligned}$$

Hence, by the identity (1) and the Lemma 2.6 we have

$$\begin{aligned} & M(c_{11}^{n-1}\cdots c_{11}^{1}a_{12}+c_{11}^{n-1}\cdots c_{11}^{1}b_{12}) \\ & \qquad +M(c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ(c_{11}^{1}\circ c_{11}^{1}))\cdots)) \\ = \; & M(c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ((a_{12}+c_{11}^{1})\circ(b_{12}+c_{11}^{1})))\cdots)) \\ = \; & M(M^{*}M^{*-1}(c_{11}^{n-1})\circ(\cdots(M^{*}M^{*-1}(c_{11}^{2}) \\ & \qquad \circ(M^{*}M^{*-1}(a_{12}+c_{11}^{1})\circ(b_{12}+c_{11}^{1})))\cdots)) \\ = \; & M^{*-1}(c_{11}^{n-1})\circ(\cdots(M^{*-1}(c_{11}^{2})\circ(M^{*-1}(a_{12}+c_{11}^{1})\circ M(b_{12}+c_{11}^{1})))\cdots) \\ = \; & M^{*-1}(c_{11}^{n-1})\circ(\cdots(M^{*-1}(c_{11}^{2})\circ((M^{*-1}(a_{12})+M^{*-1}(c_{11}^{1})) \\ & \qquad \circ(M(b_{12})+M(c_{11}^{1}))))\cdots) \\ = \; & M^{*-1}(c_{11}^{n-1})\circ(\cdots(M^{*-1}(c_{11}^{2})\circ(M^{*-1}(a_{12})\circ M(b_{12})))\cdots) \\ & +M^{*-1}(c_{11}^{n-1})\circ(\cdots(M^{*-1}(c_{11}^{2})\circ(M^{*-1}(a_{12})\circ M(c_{11}^{1})))\cdots) \\ & +M^{*-1}(c_{11}^{n-1})\circ(\cdots(M^{*-1}(c_{11}^{2})\circ(M^{*-1}(c_{11}^{1})\circ M(b_{12})))\cdots) \\ & +M^{*-1}(c_{11}^{n-1})\circ(\cdots(M^{*-1}(c_{11}^{2})\circ(M^{*-1}(c_{11}^{1})\circ M(c_{11}^{1})))\cdots) \\ = \; & M(c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ(a_{12}\circ b_{12}))\cdots)) \\ & \qquad +M(c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ(a_{12}\circ c_{11}^{1}))\cdots)) \\ & +M(c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ(c_{11}^{1}\circ b_{12}))\cdots)) \\ & \qquad +M(c_{11}^{n-1}\circ(\cdots(c_{11}^{2}\circ(c_{11}^{1}\circ c_{11}^{1}))\cdots)) \\ = \; & M(c_{11}^{n-1}\cdots c_{11}^{1}a_{12})+M(c_{11}^{n-1}\cdots c_{11}^{1}b_{12})) \end{aligned}$$

$$+M(c_{11}^{n-1} \circ (\cdots (c_{11}^{2} \circ (c_{11}^{1} \circ c_{11}^{1})) \cdots)).$$

Thus, we conclude that
$M(c_{11}^{n-1} \cdots c_{11}^{1} b_{12} + c_{11}^{n-1} \cdots c_{11}^{1} a_{12}) = M(c_{11}^{n-1} \cdots c_{11}^{1} a_{12}) + M(c_{11}^{n-1} \cdots c_{11}^{1} b_{12}))$.
(ii) We also note that the following identity is valid

$$\begin{aligned} c_{11}^{n-1} \cdots c_{11}^{2} a_{12} d_{22}^{1} + c_{11}^{n-1} \cdots c_{11}^{2} b_{12} d_{22}^{1} \\ = c_{11}^{n-1} \circ (\cdots (c_{11}^{2} \circ ((a_{12} + d_{22}^{1}) \circ (b_{12} + d_{22}^{1}))) \cdots). \end{aligned}$$

Hence, by the identity (1) and the Lemma 2.7 we have

$$\begin{aligned} & M(c_{11}^{n-1} \cdots c_{11}^{2} b_{12} d_{22}^{1} + c_{11}^{n-1} \cdots c_{11}^{2} a_{12} d_{22}^{1}) \\ = \; & M(c_{11}^{n-1} \circ (\cdots (c_{11}^{2} \circ ((a_{12} + d_{22}^{1}) \circ (b_{12} + d_{22}^{1}))) \cdots)) \\ = \; & M(M^{*}M^{*-1}(c_{11}^{n-1}) \circ (\cdots (M^{*}M^{*-1}(c_{11}^{2}) \\ & \qquad \circ (M^{*}M^{*-1}(a_{12} + d_{22}^{1}) \circ (b_{12} + d_{22}^{1}))) \cdots)) \\ = \; & M^{*-1}(c_{11}^{n-1}) \circ (\cdots (M^{*-1}(c_{11}^{2}) \circ (M^{*-1}(a_{12} + d_{22}^{1}) \circ M(b_{12} + d_{22}^{1}))) \cdots) \\ = \; & M^{*-1}(c_{11}^{n-1}) \circ (\cdots (M^{*-1}(c_{11}^{2}) \circ ((M^{*-1}(a_{12}) + M^{*-1}(d_{22}^{1})) \\ & \qquad \circ (M(b_{12}) + M(d_{22}^{1})))) \cdots) \\ = \; & M^{*-1}(c_{11}^{n-1}) \circ (\cdots (M^{*-1}(c_{11}^{2}) \circ (M^{*-1}(a_{12}) \circ M(b_{12}))) \cdots) \\ & + M^{*-1}(c_{11}^{n-1}) \circ (\cdots (M^{*-1}(c_{11}^{2}) \circ (M^{*-1}(a_{12}) \circ M(d_{22}^{1}))) \cdots) \\ & + M^{*-1}(c_{11}^{n-1}) \circ (\cdots (M^{*-1}(c_{11}^{2}) \circ (M^{*-1}(d_{22}^{1}) \circ M(b_{12}))) \cdots) \\ & + M^{*-1}(c_{11}^{n-1}) \circ (\cdots (M^{*-1}(c_{11}^{2}) \circ (M^{*-1}(d_{22}^{1}) \circ M(d_{22}^{1}))) \cdots) \\ = \; & M(c_{11}^{n-1} \circ (\cdots (c_{11}^{2} \circ (a_{12} \circ b_{12})) \cdots)) \\ & \qquad + M(c_{11}^{n-1} \circ (\cdots (c_{11}^{2} \circ (a_{12} \circ d_{22}^{1})) \cdots)) \\ & + M(c_{11}^{n-1} \circ (\cdots (c_{11}^{2} \circ (d_{22}^{1} \circ b_{12})) \cdots)) \\ & \qquad + M(c_{11}^{n-1} \circ (\cdots (c_{11}^{2} \circ (d_{22}^{1} \circ d_{22}^{1})) \cdots)) \\ = \; & M(c_{11}^{n-1} \cdots c_{11}^{2} a_{12} d_{22}^{1}) + M(c_{11}^{n-1} \cdots c_{11}^{2} b_{12} d_{22}^{1}). \end{aligned}$$

(iii) Using a similar argument to the case (i), then the identity in (iii) can be deduced from the identity

$$\begin{aligned} & a_{12} d_{22}^{1} \cdots d_{22}^{m-1} + b_{12} d_{22}^{1} \cdots d_{22}^{m-1} + d_{22}^{m-1} \circ (\cdots (d_{22}^{2} \circ (d_{22}^{1} \circ d_{22}^{1})) \cdots) \\ = \; & d_{22}^{m-1} \circ (\cdots (d_{22}^{2} \circ ((a_{12} + d_{22}^{1}) \circ (b_{12} + d_{22}^{1}))) \cdots). \end{aligned}$$

□

Lemma 2.9. *For arbitrary elements* $a_{12}, b_{12} \in \mathfrak{T}_{12}$ *the following holds:* $M(a_{12}+b_{12}) = M(a_{12}) + M(b_{12})$.

Proof. From the bijectivity of M, there is a single element $c = c_{11} + c_{12} + c_{22} \in \mathfrak{T}$ such that $M(c) = M(a_{12}) + M(b_{12})$. Hence, for arbitrary elements $c^k_{i_k j_k} \in \mathfrak{T}_{i_k j_k}$ $(1 \le i_k \le j_k \le 2; k = 2, \cdots, n-1)$, with $(i_k, j_k) = (1,1)$ $(k = 2, \cdots, n-1)$ and $d^1_{22} \in \mathfrak{T}_{22}$ we have, from Lemma 2.4 and Lemma 2.8(ii)

$$\begin{aligned} & M(c^{n-1}_{11} \circ (\cdots (c^2_{11} \circ (d^1_{22} \circ c)) \cdots)) \\ = \; & M(c^{n-1}_{11} \circ (\cdots (c^2_{11} \circ (d^1_{22} \circ a_{12})) \cdots)) \\ & + M(c^{n-1}_{11} \circ (\cdots (c^2_{11} \circ (d^1_{22} \circ b_{12})) \cdots)) \\ = \; & M(c^{n-1}_{11} \cdots c^2_{11} a_{12} d^1_{22}) + M(c^{n-1}_{11} \cdots c^2_{11} b_{12} d^1_{22}) \\ = \; & M(c^{n-1}_{11} \cdots c^2_{11} (a_{12} + b_{12}) d^1_{22}). \end{aligned}$$

It follows that $c^{n-1}_{11} \circ (\cdots (c^2_{11} \circ (d^1_{22} \circ c)) \cdots) = c^{n-1}_{11} \cdots c^2_{11} (a_{12} + b_{12}) d^1_{22}$ which implies $c^{n-1}_{11} \cdots c^2_{11} c_{12} d^1_{22} = c^{n-1}_{11} \cdots c^2_{11} (a_{12} + b_{12}) d^1_{22}$. Thus $c_{12} = a_{12} + b_{12}$, by condition (iii) of the Theorem. Also, for arbitrary elements $d^k_{i_k j_k} \in \mathfrak{T}_{i_k j_k}$ $(1 \le i_k \le j_k \le 2; k = 1, \cdots, n-1)$, with $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-1)$ we have, from Lemma 2.4 and Lemma 2.8(iii), that

$$\begin{aligned} & M(d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ c) \cdots)) \\ = \; & M(d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ a_{12}) \cdots)) + M(d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ b_{12}) \cdots)) \\ = \; & M(a_{12} d^1_{22} \cdots d^{n-1}_{22}) + M(b_{12} d^1_{22} \cdots d^{n-1}_{22}) \\ = \; & M((a_{12} + b_{12}) d^1_{22} \cdots d^{n-1}_{22}). \end{aligned}$$

This results that $d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ c) \cdots) = (a_{12} + b_{12}) d^1_{22} \cdots d^{n-1}_{22}$ which yields $d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ c_{22}) \cdots) = 0$. Thus $c_{22} = 0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $a^k_{i_k j_k} \in \mathfrak{T}_{i_k j_k}$ $(1 \le i_k \le j_k \le 2; k = 1, \cdots, n-1)$, with $(i_k, j_k) = (1,1)$ $(k = 1, \cdots, n-1)$ we have, from Lemma 2.4 and Lemma 2.8(i)

$$\begin{aligned} & M(a^{n-1}_{11} \circ (\cdots (a^1_{11} \circ c) \cdots)) \\ = \; & M(a^{n-1}_{11} \circ (\cdots (a^1_{11} \circ a_{12}) \cdots)) + M(a^{n-1}_{11} \circ (\cdots (a^1_{11} \circ b_{12}) \cdots)) \\ = \; & M(a^{n-1}_{11} \cdots a^1_{11} a_{12}) + M(a^{n-1}_{11} \cdots a^1_{11} b_{12}) \end{aligned}$$

$$= M(a_{11}^{n-1} \cdots a_{11}^{1}(a_{12} + b_{12})).$$

This implies that $a_{11}^{n-1} \circ (\cdots (a_{11}^{1} \circ c) \cdots) = a_{11}^{1} \cdots a_{11}^{n-1}(a_{12} + b_{12})$ which results $a_{11}^{n-1} \circ (\cdots (a_{11}^{1} \circ c_{11}) \cdots) = 0$. Thus $c_{11} = 0$, by condition (i) of the Theorem. □

Lemma 2.10. *For arbitrary elements $a_{11}, b_{11} \in \mathfrak{T}_{11}$ and $a_{22}, b_{22} \in \mathfrak{T}_{22}$ the following hold:*

(i) $M(a_{11} + b_{11}) = M(a_{11}) + M(b_{11})$;

(ii) $M(a_{22} + b_{22}) = M(a_{22}) + M(b_{22})$.

Proof. By the surjectivity of M there exists a single element $c = c_{11} + c_{12} + c_{22} \in \mathfrak{T}$ such that $M(c) = M(a_{11}) + M(b_{11})$. Hence, for arbitrary elements $a_{i_k j_k}^{k} \in \mathfrak{T}_{i_k j_k}$ $(1 \leq i_k \leq j_k \leq 2; k = 2, \cdots, n)$, with $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n)$ we have by Lemma 2.4

$$\begin{aligned} & M(a_{22}^{n} \circ (\cdots (a_{22}^{2} \circ c) \cdots)) \\ = \; & M(a_{22}^{n} \circ (\cdots (a_{22}^{2} \circ a_{11}) \cdots)) + M(a_{22}^{n} \circ (\cdots (a_{22}^{2} \circ b_{11}) \cdots)) \\ = \; & 0. \end{aligned}$$

This results that $a_{22}^{n} \circ (\cdots (a_{22}^{2} \circ c) \cdots) = 0$ which implies $a_{22}^{n} \circ (\cdots (a_{22}^{2} \circ c_{22}) \cdots) = 0$ and $c_{12} a_{22}^{2} \cdots a_{22}^{n} = 0$. Thus $c_{22} = 0$ and $c_{12} = 0$, by conditions (ii) and (iii), respectively, of the Theorem. Also, for arbitrary elements $a_{i_k j_k}^{k} \in \mathfrak{T}_{i_k j_k}$ $(1 \leq i_k \leq j_k \leq 2; k = 2, \cdots, n)$, with $(i_2, j_2) = (1,2)$ and $(i_k, j_k) = (2,2)$ $(k = 3, \cdots, n)$ we have by Lemma 2.4 and Lemma 2.9, that

$$\begin{aligned} & M(a_{22}^{n} \circ (\cdots (a_{22}^{3} \circ (a_{12}^{2} \circ c)) \cdots)) \\ = \; & M(a_{22}^{n} \circ (\cdots (a_{22}^{3} \circ (a_{12}^{2} \circ a_{11})) \cdots)) \\ & \qquad + M(a_{22}^{n} \circ (\cdots (a_{22}^{3} \circ (a_{12}^{2} \circ b_{11})) \cdots)) \\ = \; & M(a_{11} a_{12}^{2} a_{22}^{3} \cdots a_{22}^{n}) + M(b_{11} a_{12}^{2} a_{22}^{3} \cdots a_{22}^{n}) \\ = \; & M((a_{11} + b_{11}) a_{12}^{2} a_{22}^{3} \cdots a_{22}^{n}). \end{aligned}$$

This implies that $a_{22}^{n} \circ (\cdots (a_{22}^{3} \circ (a_{12}^{2} \circ c)) \cdots) = (a_{11} + b_{11}) a_{12}^{2} a_{22}^{3} \cdots a_{22}^{n}$ resulting in $c_{11} a_{12}^{2} a_{22}^{2} \cdots a_{22}^{n} = (a_{11} + b_{11}) a_{12}^{2} a_{22}^{3} \cdots a_{22}^{n}$. This allows us to conclude that $c_{11} = a_{11} + c_{11}$, by condition (iii) of the Theorem and the fact that $\mathfrak{T}$ is a faithfull bimodule.

Similarly we prove (ii). □

Lemma 2.11. *For arbitrary elements $a_{11} \in \mathfrak{T}_{11}$, $b_{12} \in \mathfrak{T}_{12}$ and $c_{22} \in \mathfrak{T}_{22}$ the following holds: $M(a_{11} + b_{12} + c_{22}) = M(a_{11}) + M(b_{12}) + M(c_{22})$.*

Proof. By the bijectivity of M there exists a single element $f = f_{11} + f_{12} + f_{22} \in \mathfrak{T}$ such that $M(f) = M(a_{11}) + M(b_{12}) + M(c_{22})$. Let us write $M(f) = M(a_{11} + b_{12}) + M(c_{22})$, by Lemma 2.6(i). Hence, for arbitrary elements $d^k_{i_k j_k} \in \mathfrak{T}_{i_k j_k}$ $(1 \leq i_k \leq j_k \leq 2; k = 1, \cdots, n-1)$, with $(i_k, j_k) = (1,1)$ $(k = 1, \cdots, n-1)$ we have from Lemma 2.4

$$\begin{aligned} & M(d^{n-1}_{11} \circ (\cdots (d^1_{11} \circ f) \cdots)) \\ = \; & M(d^{n-1}_{11} \circ (\cdots (d^1_{11} \circ (a_{11} + b_{12})) \cdots)) + M(d^{n-1}_{11} \circ (\cdots (d^1_{11} \circ c_{22}) \cdots)) \\ = \; & M(d^{n-1}_{11} \circ (\cdots (d^1_{11} \circ (a_{11} + b_{12})) \cdots)) \end{aligned}$$

which implies $d^{n-1}_{11} \circ (\cdots (d^1_{11} \circ f) \cdots) = d^{n-1}_{11} \circ (\cdots (d^1_{11} \circ (a_{11} + b_{12})) \cdots)$ and resulting in $d^{n-1}_{11} \circ (\cdots (d^1_{11} \circ (f_{11} - a_{11})) \cdots) = 0$ and $d^{n-1}_{11} \cdots d^1_{11}(f_{12} - b_{12}) = 0$. Thus $f_{11} = a_{11}$ and $f_{12} = b_{12}$ by conditions (i) and (iii), respectively, of the Theorem. Now, let us write $M(f) = M(a_{11}) + M(b_{12} + c_{22})$, by Lemma 2.7(i). Hence, for arbitrary elements $d^k_{i_k j_k} \in \mathfrak{T}_{i_k j_k}$ $(1 \leq i_k \leq j_k \leq 2; k = 1, \cdots, n-1)$, with $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-1)$ we have from Lemma 2.4

$$\begin{aligned} & M(d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ f) \cdots)) \\ = \; & M(d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ a_{11}) \cdots)) + M(d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ (b_{12} + c_{22})) \cdots)) \\ = \; & M(d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ (b_{12} + c_{22})) \cdots)) \end{aligned}$$

which results $d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ f) \cdots) = d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ (b_{12} + c_{22})) \cdots)$ and yielding in $d^{n-1}_{22} \circ (\cdots (d^1_{22} \circ (f_{22} - c_{22})) \cdots) = 0$. Thus $f_{22} = c_{22}$, by condition (ii) of the Theorem. □

Now we are able to prove the Theorem 2.1. Our proof is similar to that in [3].

Proof of Theorem 2.1.

Proof. We first show that M is additive. Let $a = a_{11} + a_{12} + a_{22}$ and $b = b_{11} + b_{12} + b_{22}$ be two arbitrary elements of $\mathfrak{T}$. From Lemmas 2.9, 2.10 and 2.11 we have

$$M(a + b)$$

$$
\begin{aligned}
&= M\big((a_{11}+b_{11})+(a_{12}+b_{12})+(a_{22}+b_{22})\big)\\
&= M(a_{11}+b_{11})+M(a_{12}+b_{12})+M(a_{22}+b_{22})\\
&= M(a_{11})+M(b_{11})+M(a_{12})+M(b_{12})+M(a_{22})+M(b_{22})\\
&= M(a_{11}+a_{12}+a_{22})+M(b_{11}+b_{12}+b_{22})\\
&= M(a)+M(b).
\end{aligned}
$$

Thus, M is additive. Now, to prove that M^* is additive note that, for arbitrary elements $x, y \in \mathfrak{S}$, choose c and d in $\mathfrak{T}$ such that $c = M^*(x) + M^*(y)$ and $d = M^*(x+y)$. For arbitrary elements $a^k_{i_kj_k} \in \mathfrak{T}_{i_kj_k}$ $(1 \le i_k \le j_k \le 2; k = 2, \cdots, n)$, using the additivity of M, we obtain

$$
\begin{aligned}
& M(a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ (c \circ a^2_{i_2j_2}))\cdots))\\
= \; & M(a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ ((M^*(x)+M^*(y)) \circ a^2_{i_2j_2}))\cdots))\\
= \; & M(a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ (M^*(x) \circ a^2_{i_2j_2}))\cdots)\\
& \qquad +a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ (M^*(y) \circ a^2_{i_2j_2}))\cdots))\\
= \; & M(a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ (M^*(x) \circ a^2_{i_2j_2}))\cdots))\\
& \qquad +M(a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ (M^*(y) \circ a^2_{i_2j_2}))\cdots))\\
= \; & M(M^*M^{*-1}(a^n_{in_jn}) \circ (\cdots (M^*M^{*-1}(a^3_{i_3j_3}) \circ (M^*(x) \circ a^2_{i_2j_2}))\cdots))\\
& +M(M^*M^{*-1}(a^n_{in_jn}) \circ (\cdots (M^*M^{*-1}(a^3_{i_3j_3}) \circ (M^*(y) \circ a^2_{i_2j_2}))\cdots))\\
= \; & M^{*-1}(a^n_{in_jn}) \circ (\cdots (M^{*-1}(a^3_{i_3j_3}) \circ (x \circ M(a^2_{i_2j_2})))\cdots)\\
& +M^{*-1}(a^n_{in_jn}) \circ (\cdots (M^{*-1}(a^3_{i_3j_3}) \circ (y \circ M(a^2_{i_2j_2})))\cdots)\\
= \; & M^{*-1}(a^n_{in_jn}) \circ (\cdots (M^{*-1}(a^3_{i_3j_3}) \circ ((x+y) \circ M(a^2_{i_2j_2})))\cdots)\\
= \; & M(a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ (M^*(x+y) \circ a^2_{i_2j_2}))\cdots))\\
= \; & M(a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ (d \circ a^2_{i_2j_2}))\cdots))
\end{aligned}
$$

Therefore,

$$
a^n_{in_jn} \circ (\cdots (a^3_{i_3j_3} \circ ((c-d) \circ a^2_{i_2j_2}))\cdots) = 0. \tag{5}
$$

Using a similar argument, in the identity in (5), as used in the identity (4), we can show that $c = d$, that is, $M^*(x+y) = M^*(x) + M^*(y)$.

The proof is complete. □

The ideas that follow below are similar those presented by Peisheng [3].

Suppose that $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{S}$ are algebras over the real field, or the complex field. Let $\mathfrak{T} = \mathrm{Tri}(\mathfrak{A}, \mathfrak{X}, \mathfrak{B})$ be the triangular algebra, r is a non-zero scalar and $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$ two maps. We call the ordered pair (M, M^*) a *r-Jordan n-tuple map of $\mathfrak{T} \times \mathfrak{S}$* if

$$M(r(M^*(x_n) \circ (\cdots (M^*(x_2) \circ a_1) \cdots))) = r(x_n \circ (\cdots (x_2 \circ M(a_1)) \cdots)),$$

$$M^*(r(M(a_n) \circ (\cdots (M(a_2) \circ x_1) \cdots))) = r(a_n \circ (\cdots (a_2 \circ M^*(x_1)) \cdots)),$$

for all $a_1, \cdots, a_n \in \mathfrak{T}$ and $x_1, \cdots, x_n \in \mathfrak{S}$.

We say that a r-Jordan n-tuple map (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$ is *additive* (resp., *injective, surjective, bijective*) if both maps M and M^* are additive (resp., injective, surjective, bijective). Obviously, the 1-Jordan n-tuple maps (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$ are the already known Jordan n-tuple maps (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$.

We remark that with a slightly modification, the Theorem 2.1 can be proved to be still true if the Jordan n-tuple map (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$ in Theorem 2.1 is replaced by a r-Jordan n-tuple map (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$.

Let $\mathfrak{H}$ be a Hilbert space over the field $\mathbb{F}$ of real or complex numbers. Denote by $\mathfrak{B}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$ and by I the identity operator on $\mathfrak{H}$. A chain $\mathfrak{N}$ of projections on $\mathfrak{H}$ is called a *nest* if it contains 0 and I, and it is closed in the strong operator topology. The nest algebra denoted by $\mathfrak{T}(\mathfrak{N})$ corresponding to the $\mathfrak{N}$ is defined by

$$\mathfrak{T}(\mathfrak{N}) = \{t \in \mathfrak{B}(\mathfrak{H}) \ : \ te = ete \text{ for all } e \in \mathfrak{N}\}.$$

We say that $\mathfrak{N}$ is trivial if $\mathfrak{N} = \{0, I\}$. The algebra of all finite rank operators in $\mathfrak{T}(\mathfrak{N})$ is denoted by $\mathfrak{F}(\mathfrak{N})$. The Erdos Density Theorem says that $\mathfrak{F}(\mathfrak{N})$ is dense in $\mathfrak{T}(\mathfrak{N})$ under the strong operator topology. A subalgebra of $\mathfrak{T}(\mathfrak{N})$ is called a standard subalgebra if it contains $\mathfrak{F}(\mathfrak{N})$. In particular, $\mathfrak{F}(\mathfrak{N})$ is a standard subalgebra.

Corollary 2.1. *Let $\mathfrak{N}$ be a non-trivial nest on a Hilbert space and $\mathfrak{T}$ be a standard subalgebra of the nest algebra $\mathfrak{T}(\mathfrak{N})$. If $\mathfrak{S}$ is an algebra over the field $\mathbb{F}$, then every surjective Jordan n-tuple map (M, M^*) of $\mathfrak{T} \times \mathfrak{S}$ is additive.*

Proof. First, fix an element $e \in \mathfrak{N}$ with $0 < e < I$. Set $\mathfrak{A} = e\mathfrak{T}e$, $\mathfrak{X} = e\mathfrak{T}(I-e)$ and $\mathfrak{B} = (I-e)\mathfrak{T}(I-e)$. Then we have $\mathfrak{T} = \mathrm{Tri}(\mathfrak{A}, \mathfrak{X}, \mathfrak{B})$. Since $\mathfrak{F}(\mathfrak{N})$ is dense in $\mathfrak{T}(\mathfrak{N})$ under the strong operator topology, let us consider a net $\{a_\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{F}(\mathfrak{N})$ such that $SOT - \lim_\lambda a_\lambda = I$. If $a \in \mathfrak{T}_{11}$ is such that $a_{11} \circ (\cdots(a_{11} \circ a)\cdots) = 0$ for all $a_{11} \in \mathfrak{T}_{11}$, then the limit in $(ea_\lambda e) \circ (\cdots((ea_\lambda e) \circ a)\cdots) = 0$, leads us to $a = 0$. Thus, we prove the condition (i) of the Theorem 2.1. The proof of (ii) is similar. Finally, [2, Lemma 2] allow us to conclude that $\mathfrak{X}$ is a faithful $(\mathfrak{A}, \mathfrak{B})$-bimodule satisfying the condition (iii), of the Theorem 2.1. □

Corollary 2.2. *Let $\mathfrak{N}$ be a non-trivial nest on a Hilbert space. Suppose that $\mathfrak{T}$ is a standard subalgebra of the nest algebra $\mathfrak{T}(\mathfrak{N})$, $\mathfrak{S}$ is an algebra over the field $\mathbb{F}$ and r is a non-zero scalar. Then every bijective map $\phi : \mathfrak{T} \to \mathfrak{S}$ satisfying*

$$\phi(r(a_n \circ (\cdots(a_2 \circ a_1)\cdots))) = r(\phi(a_n) \circ (\cdots(\phi(a_2) \circ \phi(a_1))\cdots))$$

for all $a_1, a_2, \cdots, a_n \in \mathfrak{T}$, is additive.

Proof. In fact, the pair (ϕ, ϕ^{-1}) is a r-Jordan n-tuple map of $\mathfrak{T} \times \mathfrak{S}$. □

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AUTOMORPHISMS OF A LOCALLY AFFINE ROOT SYSTEM OF TYPE $A^{(1)}_{\mathfrak{I}-1}$

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Abstract

We determine the automorphism group of a locally affine root system of type $A^{(1)}_{\mathfrak{I}-1}$, which particularly gives the corresponding outer automorphism group. As a corollary, the automorphism group of an affine root system of type $A^{(1)}_{n-1}$ is well understood in a certain general picture.

Keywords: affine root system; locally affine root system; Weyl group.

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1 Introduction

Let us start with the definition of locally extended affine root systems introduced in [7] (see also [14]).

Definition 1.1 Let V be a vector space over $\mathbb{Q}$ with a positive semidefinite bilinear form $(\cdot,\cdot)$. A subset R of V is called a **locally extended affine root system** or a **LEARS** for short if

(A1) $(\alpha,\alpha) \neq 0$ for all $\alpha \in R$, and R spans V;

(A2) $\langle\alpha,\beta\rangle \in \mathbb{Z}$ for all $\alpha,\beta \in R$, where $\langle\alpha,\beta\rangle = \dfrac{2(\alpha,\beta)}{(\beta,\beta)}$;

(A3) $\sigma_\alpha(\beta) \in R$ for all $\alpha,\beta \in R$, where $\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$;

(A4) $R = R_1 \cup R_2$ and $(R_1,R_2) = 0$ imply $R_1 = \emptyset$ or $R_2 = \emptyset$. (R is irreducible.)

A LEARS R is called **reduced** if $2\alpha \notin R$ for all $\alpha \in R$.

Definition 1.2 LEARS (V,R) and (V',R') are called **isomorphic**, denoted by $R \cong R'$, if there is a linear isomorphism $\varphi : V \longrightarrow V'$ such that $\varphi(R) = R'$ and $\langle\alpha,\beta\rangle = \langle\varphi(\alpha),\varphi(\beta)\rangle$ for all $\alpha,\beta \in R$. In particular, we define the automorphism group, called $\mathrm{Aut}\,R$, of a LEARS R by $\mathrm{Aut}\,R = \{\varphi \mid \varphi : R \xrightarrow{\sim} R \text{ an isomorphism}\}$.

We note that $\sigma_\alpha \in \mathrm{Aut}\,R$ and $-1_V \in \mathrm{Aut}\,R$. Then, we define $W(R)$ to be the subgroup, called the **Weyl group**, of $\mathrm{Aut}\,R$ generated by σ_α for all $\alpha \in R$. Since $\varphi \circ \sigma_\alpha \circ \varphi^{-1} = \sigma_{\varphi(\alpha)}$, the Weyl group $W(R)$ is a normal subgroup of $\mathrm{Aut}\,R$, which allows us to put $\mathrm{Out}\,R = \mathrm{Aut}\,R/W(R)$ as the outer automorphism group of R.

Let $V^0 := \{x \in V \mid (x,y) = 0 \text{ for all } y \in V\}$ be the radical of the form. We call a LEARS (R,V) an **extended affine root system** or an **EARS** for short, if $\dim_{\mathbb{Q}} V/V^0 < \infty$ and $\langle R\rangle$ is free. This coincides with the concept, which was firstly introduced by Saito in 1985 [12]. The notion of an EARS was also used in a different sense in [2], but Azam showed that there is a natural correspondence between the two notions in [1]. EARS in

Saito's sense naturally generalize the Macdonald's **affine root systems** in [6].

When the torsion-free abelian group $\langle R\rangle \cap V^0$ is free, we say that R has **nullity**. Our LEARS are a natural generalization of the Saito's EARS. In fact, Saito's EARS are the same as our EARS embedded into the real vector space $\mathbb{R}\otimes_{\mathbb{Q}} V$. Similarly, irreducible affine root systems in the sense of Macdonald [6] are our EARS of nullity 1. Note that the reduced irreducible affine root systems are the real roots of affine Kac-Moody Lie algebras. The elliptic root systems defined by Saito [12] are our EARS of nullity 2. Also, the sets of nonisotropic roots of EARS in [2] are our reduced EARS of finite rank (see [1]).

We call a LEARS of nullity 1 a **locally affine root system**, which generalizes irreducible affine root systems (see [10], [8]). Note that LEARS of nullity 0 are so-called locally finite irreducible root systems, which generalizes finite irreducible root systems (see [11], [13]).

For example, let

$$V^\sharp = \oplus_{i\in\mathfrak{I}} \mathbb{Q}\epsilon_i$$

be a vector space over $\mathbb{Q}$, where $\mathfrak{I}$ is any index set, and $\{\epsilon_i \mid i \in \mathfrak{I}\}$ is an orthonormal basis of $V^\sharp$. Then $R = \{\epsilon_i - \epsilon_j \mid i \neq j \in \mathfrak{I}\}$ in

$$V := \oplus_{j\in\mathfrak{I}\setminus\{1\}} \mathbb{Q}(\epsilon_1 - \epsilon_j) \subset V^\sharp$$

is a locally finite root system of type $\mathrm{A}_{\mathfrak{I}-1}$, where $1 \in \mathfrak{I}$ is a fixed element. Also,

$$R = \{\epsilon_i - \epsilon_j + k\delta \mid i \neq j \in \mathfrak{I},\ k \in \mathbb{Z}\}$$

in $\tilde{V} = V \oplus \mathbb{Q}\delta$ with $\tilde{V}^0 = \mathbb{Q}\delta$ is a locally affine root system of type $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$. In this paper, we concentrate to determine the automorphism group of a locally affine root system of type $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$. As a corollary, the automorphism group of an affine root system of type $\mathrm{A}^{(1)}_{n-1}$ is well understood in a certain general picture. The result, that is, the structure of the automorphism group of type $\mathrm{A}^{(1)}_{n-1}$, was shown as a corollary of the conjugacy theorem of root bases (see e.g. [4, Prop.5.5, Cor.5.10] or [3, Thm.3.1]), but we show this directly as a special case of type $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$ in an elementary way.

A locally extended affine root system is the set of anisotropic roots of a **locally affine Lie algebra** with a fixed maximal ad-diagonalizable

subalgebra (see [8]). Therefore, it is very important to study those roots to establish the structure theorems as well as the classification theorems of the corresponding Lie algebras. Especially, to determine the automorphism of our locally affine root system of type $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$ is very useful to obtain the classification theorems in [9]. In this paper, we will discuss this group as elementary as possible only using some set theoretical approaches. The most difficult point is that the cardinality of $\mathfrak{I}$ is infinite. We believe that the automorphism groups of locally affine root systems for other types will be also important in future research.

2 Automorphisms of type $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$

Definition 2.1 Let $\mathfrak{I}$ be any index set. We denote $S_{\mathfrak{I}}$ the symmetric group of $\mathfrak{I}$, $S_{(\mathfrak{I})}$ the subgroup of $S_{\mathfrak{I}}$ generated by the transpositions, $\mathbb{Z}^{\mathfrak{I}}$ the ($|\mathfrak{I}|$ times) direct product of $\mathbb{Z}$, and $\mathbb{Z}^{(\mathfrak{I})}$ the direct sum of $\mathbb{Z}$. Also, for $x = (m_i)_{i\in\mathfrak{I}} \in \mathbb{Z}^{(\mathfrak{I})}$, we write $\mathrm{tr}(x) = \sum_{i\in\mathfrak{I}} m_i$, and it is called the **trace** of x.

Theorem 2.2 *Let $R = \{\epsilon_i - \epsilon_j + k\delta \mid i \neq j \in \mathfrak{I},\ k \in \mathbb{Z}\}$ be a locally affine root system of type $A^{(1)}_{\mathfrak{I}-1}$. Let φ be an automorphism of R. Then $\varphi(\delta) = \xi\delta$, where $\xi = \pm 1$, and there exists some permutation $\sigma \in S_{\mathfrak{I}}$ such that $\varphi(\epsilon_i - \epsilon_j) = \eta(\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + m_{ij}\delta)$ for some $m_{ij} \in \mathbb{Z}$, where $\eta = \pm 1$. Moreover, we have*

$$\begin{aligned} Aut\, R &\cong \{\pm 1_V\} \times \left((S_{\mathfrak{I}} \times \{\pm 1\}_\eta) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\right) \cong \mathbb{Z}_2 \times \left((\mathbb{Z}_2 \times S_{\mathfrak{I}}) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\right) \\ &\cong \{\pm 1_V\} \times \left((S_{\mathfrak{I}} \times \{\pm 1\}_\eta) \ltimes (\mathbb{Z}^{\mathfrak{I}}/\mathbb{Z}\iota)\right) \cong \mathbb{Z}_2 \times \left((\mathbb{Z}_2 \times S_{\mathfrak{I}}) \ltimes (\mathbb{Z}^{\mathfrak{I}}/\mathbb{Z}\iota)\right), \end{aligned}$$

where ι is the identity vector, i.e., all the coordinates of ι are 1, and the Weyl group of R is

$$W(R) \cong S_{(\mathfrak{I})} \ltimes \mathbb{Z}_0^{(\mathfrak{I})},$$

where $\mathbb{Z}_0^{(\mathfrak{I})} = \{x \in \mathbb{Z}^{(\mathfrak{I})} \mid tr(x) = 0\}$.

Proof

First, we show that $\varphi(\delta) = \pm\delta$. Since δ is in the radical of the positive semi-definite form on the vector space spanned by R over $\mathbb{Q}$, we know that

$\varphi(\delta) \in \mathbb{Q}\delta$. Let $\varphi(\delta) = \ell\delta$ for $\ell \in \mathbb{Q}$. Since $\varphi(\epsilon_i - \epsilon_j + \delta) = (\epsilon_r - \epsilon_s + m\delta) + \ell\delta \in R$ for some $r \neq s \in \mathfrak{I}$ and $m \in \mathbb{Z}$, we have $\ell \in \mathbb{Z}$. Note that $\ell \neq 0$ since φ is one to one. So we have $\frac{1}{\ell}\varphi(\delta) = \delta$. If $|\ell| > 1$, then we have $\varphi(\epsilon_i - \epsilon_j + \frac{1}{\ell}\delta) = (\epsilon_r - \epsilon_s + m\delta) + \delta \in R$. This is a contradiction since $\epsilon_i - \epsilon_j + \frac{1}{\ell}\delta \notin R$. Therefore, we get $|\ell| = 1$, and so $\varphi(\delta) = \pm\delta$.

Let $1 \in \mathfrak{I}$ be fixed. Let $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_y + m\delta$ and $\varphi(\epsilon_1 - \epsilon_j) = \epsilon_s - \epsilon_t + n\delta$ for some $x, y, s, t \in \mathfrak{I}$ and $m, n \in \mathbb{Z}$. Since $(\epsilon_1 - \epsilon_j) - (\epsilon_1 - \epsilon_i) = \epsilon_i - \epsilon_j \in R$, we have $(\epsilon_s - \epsilon_t) - (\epsilon_x - \epsilon_y) + (n-m)\delta \in R$. Hence $s = x$ or $t = y$. Suppose that $s = x$, and so we have $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_y + m\delta$ and $\varphi(\epsilon_1 - \epsilon_j) = \epsilon_x - \epsilon_t + n\delta$. Note that if $y = t$, then $\varphi(\epsilon_i - \epsilon_j) = (n-m)\delta$, which is a cotradiction. Hence, $y \neq t$. If $|\mathfrak{I}| \geq 4$, then for any $k \in \mathfrak{I} \setminus \{1, i, j\}$, by the same argument above, we have $\varphi(\epsilon_1 - \epsilon_k) = \epsilon_x - \epsilon_z + \ell\delta$ or $\epsilon_u - \epsilon_y + \ell\delta$ for some $z, u \in \mathfrak{I}$ and $\ell \in \mathbb{Z}$, comparing with $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_y + m\delta$, and $\varphi(\epsilon_1 - \epsilon_k) = \epsilon_x - \epsilon_z + \ell\delta$ or $\epsilon_v - \epsilon_t + \ell\delta$ for some $v \in \mathfrak{I}$, comparing with $\varphi(\epsilon_1 - \epsilon_j) = \epsilon_x - \epsilon_t + n\delta$. Thus, if $\varphi(\epsilon_1 - \epsilon_k) \neq \epsilon_x - \epsilon_z + \ell\delta$, then $\epsilon_u - \epsilon_y + \ell\delta = \epsilon_v - \epsilon_t + r\delta$. This forces $y = t$, which is a contradiction. Hence, we have $\varphi(\epsilon_1 - \epsilon_k) = \epsilon_x - \epsilon_z + \ell\delta$. Thus we can write $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_{\sigma(i)} + m_i\delta$ for some $m_i \in \mathbb{Z}$, where σ is a map from $\mathfrak{I} \setminus \{1\}$ to $\mathfrak{I} \setminus \{x\}$. If $\sigma(i) = \sigma(j)$, then $\varphi(\epsilon_i - \epsilon_j) = (m_j - m_i)\delta$, which is a contradiction. Hence σ is injective. For any $s \in \mathfrak{I} \setminus \{x\}$, there exist some $p, q \in \mathfrak{I}$ and $m \in \mathbb{Z}$ such that $\varphi(\epsilon_p - \epsilon_q + m\delta) = \epsilon_x - \epsilon_s$ since φ is surjective. If $q = 1$, then $\varphi(\epsilon_p - \epsilon_1 + m\delta) = -\epsilon_x + \epsilon_{\sigma(p)}$, which forces $x = s$. But then $\varphi(\epsilon_p - \epsilon_q + m\delta) = \epsilon_s - \epsilon_s = 0$, which is a contradiction. Thus we assume that $q \neq 1$. Note that $\epsilon_p - \epsilon_q = (\epsilon_p - \epsilon_1) + (\epsilon_1 - \epsilon_q)$, and so if $p \neq 1$, then $\epsilon_x - \epsilon_\ell = \varphi(\epsilon_p - \epsilon_q + m\delta) = (\epsilon_{\sigma(p)} - \epsilon_x) + (\epsilon_x - \epsilon_{\sigma(q)}) = \epsilon_{\sigma(p)} - \epsilon_{\sigma(q)}$. Hence, we have $x = \sigma(p)$, which contradicts the definition of σ. Therefore, p has to be 1, and then $\ell = \sigma(q)$. Thus we have shown the surjectivity of σ, and so σ is a bijection from $\mathfrak{I} \setminus \{1\}$ onto $\mathfrak{I} \setminus \{x\}$. So, if we define $\sigma(1) = x$, then σ is a permutation on $\mathfrak{I}$. Then, for any $i \neq j \in \mathfrak{I}$, we have $\varphi(\epsilon_i - \epsilon_j) = \varphi(\epsilon_i - \epsilon_1) + \varphi(\epsilon_1 - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + (m_j - m_i)\delta$. Let us define

$$m_1 = 0 \quad \text{and} \quad m_{ij} := m_j - m_i.$$

Then we can write

$$\varphi(\epsilon_i - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + m_{ij}\delta,$$

even though $i = 1$ or $j = 1$. So $m_{1i} = m_i$ for all $i \in \mathfrak{I} \setminus \{1\}$.

If $t = y$, then $-\varphi(\epsilon_1 - \epsilon_i) = \epsilon_y - \epsilon_x - m\delta$ and $-\varphi(\epsilon_1 - \epsilon_j) = \epsilon_y - \epsilon_s - n\delta$ for $x \neq s$. Thus, by the same argument above, we get $-\varphi(\epsilon_1 - \epsilon_i) = \epsilon_y - \epsilon_{\sigma(i)} + m_i\delta$ for some $m_i \in \mathbb{Z}$, where σ is a permutation on $\mathfrak{I}$ with $\sigma(1) = y$, and get $-\varphi(\epsilon_i - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + (m_j - m_i)\delta$, i.e.,

$$\varphi(\epsilon_i - \epsilon_j) = -(\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + m_{ij}\delta)$$

for some $\sigma \in S_{\mathfrak{I}}$ and all $i \neq j \in \mathfrak{I}$. Thus the first assertion is shown.

We have shown that, for $\varphi \in \operatorname{Aut} R$, there exists

$$(\eta, \sigma, \xi, (m_i)_{i\in\mathfrak{I}\setminus\{1\}}) = (\eta, \sigma, \xi, (m_{1i})_{i\in\mathfrak{I}\setminus\{1\}}) \in \left(\{\pm 1\}, S_{\mathfrak{I}}, \{\pm 1\}, \mathbb{Z}^{\mathfrak{I}-1}\right)$$

such that

$$\varphi : \begin{cases} \epsilon_1 - \epsilon_k \mapsto \eta(\epsilon_{\sigma(1)} - \epsilon_{\sigma(k)} + m_{1k}\delta) \\ \delta \mapsto \xi\delta\ . \end{cases}$$

On the other hand, we notice that a quadruple of these η, σ, ξ and (m_i) determines an automorphism of R conversely. Let $\varphi = (\eta, \sigma, \xi, (m_{1i})_{i\in\mathfrak{I}\setminus\{1\}})$ and $\phi' = (\eta', \sigma', \xi', (m'_{1i})_{i\in\mathfrak{I}\setminus\{1\}}) \in \left(\{\pm 1\}, S_{\mathfrak{I}}, \{\pm 1\}, \mathbb{Z}^{\mathfrak{I}-1}\right)$. Then we have

$$\varphi \circ \psi : \begin{cases} \epsilon_1 - \epsilon_k \mapsto \eta'(\epsilon_{\sigma'(1)} - \epsilon_{\sigma'(k)} + m'_{1k}\delta) \\ \mapsto \eta\eta'(\epsilon_{\sigma\sigma'(1)} - \epsilon_{\sigma\sigma'(k)} + m_{\sigma'(1)\sigma'(k)}\delta) + \xi\eta' m'_{1k}\delta \\ = \eta\eta'\left(\epsilon_{\sigma\sigma'(1)} - \epsilon_{\sigma\sigma'(k)} + (m_{\sigma'(1)\sigma'(k)} + \xi\eta m'_{1k})\delta\right) \\ \delta \mapsto \xi'\delta \mapsto \xi\xi'\delta, \end{cases}$$

and so

$$\varphi \circ \psi = \left(\eta\eta', \sigma\sigma', \xi\xi', (m_{\sigma'(1)\sigma'(i)} + \xi\eta m'_{1i})_{i\in\mathfrak{I}\setminus\{1\}})\right). \tag{1}$$

Thus through this composite, we have the group epimorphism Ψ from the group

$$(\{\pm 1\}_\eta \times S_{\mathfrak{I}} \times \{\pm 1\}_\xi) \ltimes \mathbb{Z}^{\mathfrak{I}-1}$$

onto $\operatorname{Aut} R$. If $\varphi \in \ker \Psi$, then $\varphi(\epsilon_1 - \epsilon_k) = \eta(\epsilon_{\sigma(1)} - \epsilon_{\sigma(k)} + m_{1k}\delta) = \epsilon_1 - \epsilon_k$ for all k and $\varphi(\delta) = \xi\delta = \delta$, which implies that $\eta = 1$, $\sigma = 1$, $m_k = 0$ and $\xi = 1$. Therefore, we obtain that $\ker \Psi$ is trivial, which implies

$$\operatorname{Aut} R \cong (\{\pm 1\}_\eta \times S_{\mathfrak{I}} \times \{\pm 1\}_\xi) \ltimes \mathbb{Z}^{\mathfrak{I}-1}.$$

In particular, we get the following four subgroups of $\mathrm{Aut}\, R$:

$$\begin{aligned} \{\pm 1\}_\eta &= \{(\pm 1, \mathrm{id}, 1, \mathbf{0})\}, \\ S_\mathfrak{I} &= \{(1, \sigma, 1, \mathbf{0}) \mid \sigma \in S_\mathfrak{I}\}, \\ \{\pm 1\}_\xi &= \{(1, \mathrm{id}, \pm 1, \mathbf{0})\}, \\ \mathbb{Z}^{\mathfrak{I}-1} &= \{(1, \mathrm{id}, 1, (m_i)_{i\in\mathfrak{I}-1}) \mid m_i \in \mathbb{Z}\}. \end{aligned}$$

Here, we determine the center Z of $\mathrm{Aut}\, R$. Let $\varphi = (\eta, \sigma, \xi, (m_i)_{i\in\mathfrak{I}-1})$ be a central element of $\mathrm{Aut}\, R$. Then, we obtain:

$$\begin{aligned} & \varphi \text{ is central} \\ \Leftrightarrow\ & \varphi\circ\varphi' = \varphi'\circ\varphi,\ \forall\varphi' = (\eta', \sigma', \xi', (m_i')_{i\in\mathfrak{I}-1}) \\ \Leftrightarrow\ & \eta\eta' = \eta'\eta,\ \sigma\sigma' = \sigma'\sigma,\ \xi\xi' = \xi'\xi,\ m_{\sigma'(1),\sigma'(i)} + \xi\eta m'_{1i} = m'_{\sigma(1),\sigma(i)} + \xi'\eta' m_{1i},\ \forall\varphi' \\ \Leftrightarrow\ & \sigma\sigma' = \sigma'\sigma,\ m_{\sigma'(1),\sigma'(i)} + \xi\eta m'_{1i} = m'_{\sigma(1),\sigma(i)} + \xi'\eta' m_{1i},\ \forall\varphi' \\ \Leftrightarrow\ & \sigma = \mathrm{id},\ m_{\sigma'(1),\sigma'(i)} + \xi\eta m'_{1i} = m'_{1i} + \xi'\eta' m_{1i},\ \forall\varphi' \\ \Leftrightarrow\ & \sigma = \mathrm{id},\ \xi\eta = 1,\ m_{\sigma'(1),\sigma'(i)} = \xi'\eta' m_{1i},\ \forall\varphi' \\ \Leftrightarrow\ & \sigma = \mathrm{id},\ \xi\eta = 1,\ m_{1i} = 0 \\ \Leftrightarrow\ & \varphi = (1, \mathrm{id}, 1, \mathbf{0}) \text{ or } (-1, \mathrm{id}, -1, \mathbf{0}). \end{aligned}$$

Let $\nu = (1, 1, -1, \mathbf{0})$ for $\xi = -1$ and $\pi = (-1, 1, 1, \mathbf{0})$ for $\eta = -1$. Then

$$-1_V = \nu\pi = (-1, \mathrm{id}, -1, \mathbf{0})$$

is the automorphism of the multiplication by -1, which is in the center Z of $\mathrm{Aut}\, R$. Thus we get

$$\mathrm{Aut}\, R \cong \{\pm 1_V\} \times \left((S_\mathfrak{I} \times \{\pm 1\}_\eta) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\right) \cong \mathbb{Z}_2 \times \left((S_\mathfrak{I} \times \mathbb{Z}_2) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\right).$$

From now on, we want to determine $W(R)$ and $\mathrm{Out}\, R$. To do so, we need to have some identification. To show another isomorphism, we identify $\mathbb{Z}^{\mathfrak{I}-1}$ with $\mathbb{Z}^\mathfrak{I}/\mathbb{Z}\iota$ by the group isomorphisim

$$f : (x_2, x_3, \ldots) \mapsto (0, x_2, x_3 \ldots) + \mathbb{Z}\iota = \overline{(0, x_2, x_3 \ldots)}.$$

Thus, for $\varphi = (\eta, \sigma, \xi, (m_i)_{i\in\mathfrak{I}\setminus\{1\}})$, we can write

$$\varphi = (\eta, \sigma, \xi, \overline{(m_i)}_{i\in\mathfrak{I}}) \in (\{\pm 1\}_\eta \times S_\mathfrak{I} \times \{\pm 1\}_\xi) \ltimes (\mathbb{Z}^\mathfrak{I}/\mathbb{Z}\iota),$$

and in (1), we have $(m_{\sigma'(1)\sigma'(i)}+\xi\eta m'_{1i})_{i\in\mathfrak{I}\setminus\{1\}} = (m_{\sigma'(i)}-m_{\sigma'(1)}+\xi\eta m'_i)_{i\in\mathfrak{I}\setminus\{1\}}$. Hence, we get

$$\begin{aligned} f\big((m_{\sigma'(1)\sigma'(i)} + \xi\eta m'_{1i})_{i\in\mathfrak{I}\setminus\{1\}}\big) &= \overline{(m_{\sigma'(i)} - m_{\sigma'(1)} + \xi\eta m'_i)}_{i\in\mathfrak{I}} \\ &= \overline{(m_{\sigma'(i)} + \xi\eta m'_i)}_{i\in\mathfrak{I}} \end{aligned}$$

since

$$\overline{(x_i)}_{i\in\mathfrak{I}} = \overline{(x_i + x)}_{i\in\mathfrak{I}} \quad \text{for any } x \in \mathbb{Z}$$

and the coordinate of $1 \in \mathfrak{I}$ in the last expression, i.e., $m_{\sigma'(1)} + \xi\eta m'_1$, is equal to $m_{\sigma'(1)}$ (since $m'_1 = 0$ by definition). Therefore, we obtain

$$\operatorname{Aut} R \cong (\{\pm 1\}_\eta \times S_\mathfrak{I} \times \{\pm 1\}_\xi) \ltimes (\mathbb{Z}^\mathfrak{I}/\mathbb{Z}\iota),$$

and for $\varphi = (\eta, \sigma, \xi, \overline{(m_i)}_{i\in\mathfrak{I}})$ and $\psi = (\eta', \sigma', \xi', \overline{(m'_i)}_{i\in\mathfrak{I}}), \in \operatorname{Aut} R$,

$$\varphi \circ \psi = \big(\eta\eta', \sigma\sigma', \xi\xi', \overline{(m_{\sigma'(i)} + \xi\eta m'_i)}_{i\in\mathfrak{I}}\big). \tag{2}$$

Also, by the same reason above (using $-\mathbf{1}_V = \nu\pi$), we get

$$\operatorname{Aut} R \cong \{\pm\mathbf{1}_\mathbf{V}\} \times \big((S_\mathfrak{I} \times \{\pm 1\}_\eta) \ltimes (\mathbb{Z}^\mathfrak{I}/\mathbb{Z}\iota)\big) \cong \mathbb{Z}_2 \times \big((S_\mathfrak{I} \times \mathbb{Z}_2) \ltimes (\mathbb{Z}^\mathfrak{I}/\mathbb{Z}\iota)\big).$$

The Weyl group $W(R)$ is generated by $\sigma_{\epsilon_1-\epsilon_j+r\delta}$ since

$$\{\epsilon_1 - \epsilon_j \mid j \in \mathfrak{I} \setminus \{1\}\}$$

is a reflectable basis of the locally finite root system $\mathrm{A}_{\mathfrak{I}-1}$ (see [14]). Also, we observe that $\sigma_{\epsilon_1-\epsilon_j+r\delta}(\delta) = \delta$ and

$$\sigma_{\epsilon_1-\epsilon_j+r\delta}(\epsilon_1 - \epsilon_i) = \sigma_{\epsilon_1-\epsilon_j}(\epsilon_1 - \epsilon_i) - r\langle\epsilon_1 - \epsilon_i, \epsilon_1 - \epsilon_j\rangle\delta.$$

So we have

$$\sigma_{\epsilon_1-\epsilon_j+r\delta} = (1, (1, j), 1, \overline{(m_i)}_{i\in\mathfrak{I}}) = (1, (1, j), 1, f((m_{1i})_{i\in\mathfrak{I}\setminus\{1\}})),$$

where $m_{1j} = -2r$ and $m_{1i} = -r$ for $i \neq j$, and f is defined above. Thus,

$$\begin{aligned} f((-r, \ldots, -r, -2r, -r, \ldots)) &= \overline{(0, -r, \ldots, -r, -2r, -r, \ldots)} \\ &= \overline{(r, 0, \ldots, 0, -r, 0, \ldots)}, \end{aligned}$$

where the last representative has the trivial trace. (We simply put the first coordinate to be the coordinate of $1 \in \mathfrak{I}$ for covenience.) Thus, the Weyl group $W(R)$ of R is isomorphic to $S_{(\mathfrak{I})} \ltimes \mathbb{Z}_0^{(\mathfrak{I})}$ since $(\mathbb{Z}_0^{(\mathfrak{I})} + \mathbb{Z}\iota)/\mathbb{Z}\iota \cong \mathbb{Z}_0^{(\mathfrak{I})}$. Hence, we obtain

$$\operatorname{Out} R \cong \{\pm 1_V\} \times \Big(\big(S_{\mathfrak{I}}/S_{(\mathfrak{I})} \times \mathbb{Z}_2\big) \ltimes \big(\mathbb{Z}^{\mathfrak{I}}/(\mathbb{Z}_0^{(\mathfrak{I})} + \mathbb{Z}\iota)\big) \Big). \quad \square$$

As a corollary of Theorem 2.2, we can determine the automorphism group of an affine root system of type $\mathrm{A}_{n-1}^{(1)}$ in an elementary way although the result itself is known (see e.g. [4, Cor.5.10]).

Corollary 2.3 *Let $R = \{\epsilon_i - \epsilon_j + k\delta \mid 1 \le i \ne j \le n,\ k \in \mathbb{Z}\}$ be an affine root system of type $A_{n-1}^{(1)}$. Then we have*

$$Aut\, R \cong \{\pm 1_V\} \times \big((S_n \ltimes \mathbb{Z}_0^n) \rtimes I_n\big) \cong \mathbb{Z}_2 \times \big((S_n \ltimes \mathbb{Z}_0^n) \rtimes I_n\big),$$

where $\mathbb{Z}_0^n = \{x \in \mathbb{Z}^n \mid tr(x) = 0\} \cong \mathbb{Z}^{n-1}$ and I_n is the dihedral group of order $2n$, and the Weyl group of R is $S_n \ltimes \mathbb{Z}_0^n$.

Proof

When $|\mathfrak{I}| = n$ in Theorem 2.2, we have $S_{\mathfrak{I}} = S_{(\mathfrak{I})} = S_n$ and $\operatorname{Out} R \cong \mathbb{Z}_2 \ltimes \big(\mathbb{Z}^n/(\mathbb{Z}_0^n + \mathbb{Z}\iota)\big)$. Moreover, we have:

$$\begin{aligned}
&\mathbb{Z}^n/(\mathbb{Z}_0^n + \mathbb{Z}\iota) \cong C_n = \langle s \rangle \text{ (a cyclic group)} \\
&\text{via} \quad \mathbf{x} + \mathbb{Z}_0^n + \mathbb{Z}\iota \mapsto s^{\mathrm{tr}(\mathbf{x})} \quad \text{and} \qquad (3) \\
&\mathbb{Z}_2 \ltimes \big(\mathbb{Z}^n/(\mathbb{Z}_0^n + \mathbb{Z}\iota)\big) \cong I_n = \langle s, t \rangle \text{ (a dihedral group)} \\
&\text{via} \quad \theta : (\eta, \mathbf{x} + \mathbb{Z}_0^n + \mathbb{Z}\iota) \mapsto s^{\mathrm{tr}(\mathbf{x})} t^{\varepsilon(\eta,-1)}, \qquad (4)
\end{aligned}$$

where $\varepsilon(i,j)$ means 1 (resp. 0) if $i = j$ (resp. $i \ne j$), and s and t are generators satisfying $s^n = t^2 = 1$ and $tst = s^{-1}$. In fact, the map defined in (3) is clearly a well-defined epimorphism. To show one to one, if $\mathrm{tr}(\mathbf{x}) = nk$ for some $k \in \mathbb{Z}$, then

$$\begin{aligned}
\bar{\mathbf{x}} &= \overline{\mathbf{x} + (0, \dots, 0, -\mathrm{tr}(\mathbf{x})) + (0, \dots, 0, \mathrm{tr}(\mathbf{x}))} \\
&= \overline{(0, \dots, 0, \mathrm{tr}(\mathbf{x}))} \\
&= \overline{(k, \dots, k) - (k, \dots, k, -nk + k)} \\
&= \bar{0} \mod \mathbb{Z}_0^n + \mathbb{Z}\iota,
\end{aligned}$$

and hence one to one. The map θ in (4) is clearly well-defined and surjective. That θ is a homomorphism follows from

$$\begin{aligned}\theta((\eta,\bar{\mathbf{x}})(\eta',\bar{\mathbf{x}}')) &= \theta((\eta\eta',\overline{\mathbf{x}+\eta\mathbf{x}'}))\\ &= s^{\mathrm{tr}(\mathbf{x}+\eta\mathbf{x}')}t^{\varepsilon(\eta\eta',-1)}\\ &= \begin{cases} s^{\mathrm{tr}(\mathbf{x})}s^{\mathrm{tr}(\mathbf{x}')}t^{\varepsilon(\eta',-1)} & \text{if } \eta=1\\ s^{\mathrm{tr}(\mathbf{x})}s^{-\mathrm{tr}(\mathbf{x}')}tt^{\varepsilon(-\eta',-1)} = s^{\mathrm{tr}(\mathbf{x})}ts^{\mathrm{tr}(\mathbf{x}')}t^{\varepsilon(\eta',-1)} & \text{if } \eta=-1\end{cases}\\ &= \theta((\eta,\bar{\mathbf{x}}))\theta((\eta',\bar{\mathbf{x}}'))\end{aligned}$$

for $(\eta,\bar{\mathbf{x}})$, $(\eta',\bar{\mathbf{x}}') \in \mathbb{Z}_2 \ltimes \big(\mathbb{Z}^n/(\mathbb{Z}_0^n + \mathbb{Z}\iota)\big)$. If $\theta((\eta,\bar{\mathbf{x}})) = s^{\mathrm{tr}(\mathbf{x})}t^{\varepsilon(\eta,-1)} = 1$, then $\mathrm{tr}(\mathbf{x}) = nk$ for some $k \in \mathbb{Z}$ and $\eta = 1$. Thus $\bar{\mathbf{x}} = \bar{\mathbf{0}}$ as above, and so θ is an isomorphism.

Here we notice that the following exact sequence splits.

$$0 \longrightarrow S_n \ltimes \mathbb{Z}_0^n \hookrightarrow (S_n \times \{\pm 1\}_\eta) \ltimes (\mathbb{Z}^n/\mathbb{Z}\iota) \longrightarrow I_n \longrightarrow 0$$

In fact, let $\psi : I_n \longrightarrow (S_n \times \{\pm 1\}_\eta) \ltimes (\mathbb{Z}^n/\mathbb{Z}\iota)$ by

$$\begin{aligned} s &\mapsto S = \big(1, v, 1, \overline{(0,\ldots,0,1)}\big)\\ t &\mapsto T = \big(-1, w, 1, \bar{\mathbf{0}}\big),\end{aligned}$$

where

$$\begin{aligned} &v = (1,2,\ldots,n) \quad \text{and}\\ &w = (1,n)(2,n-1)\cdots(p-1,p+2)(p,p+1) \quad \text{if } n = 2p \quad \text{or}\\ &w = (1,n)(2,n-1)\cdots(p-1,p+3)(p,p+2) \quad \text{if } n = 2p+1\end{aligned}$$

Then, we can find

$$S^n = T^2 = \big(1, \mathrm{id}, 1, \bar{\mathbf{0}}\big) \quad \text{and} \quad TST = S^{-1},$$

which is confirmed by the following direct calculation:

$$\begin{aligned}
S^n &= S^{n-2} \circ \big(1, v, 1, \overline{(0,\dots,0,0,1)}\big) \circ \big(1, v, 1, \overline{(0,\dots,0,1)}\big) \\
&= S^{n-2} \circ \big((1, v^2, 1, \overline{(0,0,\dots,0,1,1)}\big) \\
&= \cdots\cdots\cdots \\
&= S \circ \big((1, v^{n-1}, 1, \overline{(0,1,\dots,1,1,1)}\big) = \big((1, \mathrm{id}, 1, \bar{\mathbf{0}}\big), \\
T^2 &= \big((1, w^2, 1, \bar{\mathbf{0}}\big) = \big((1, \mathrm{id}, 1, \bar{\mathbf{0}}\big), \\
TST &= \big(-1, w, 1, \bar{\mathbf{0}}\big) \circ \Big(\big(1, v, 1, \overline{(0,\dots,0,1)}\big) \circ \big(-1, w, 1, \bar{\mathbf{0}}\big)\Big) \\
&= \big(-1, w, 1, \bar{\mathbf{0}}\big) \circ \big(-1, vw, 1, \overline{(0,\dots,0,1)}\big) \\
&= \big(1, wvw, 1, \overline{(0,1,\dots,1)}\big) = S^{-1}.
\end{aligned}$$

Therefore, we see that

$$\mathrm{Aut}\, R \cong \{\pm 1_V\} \ltimes \big((S_n \ltimes \mathbb{Z}_0^n) \rtimes \mathrm{I}_n\big) \cong \mathbb{Z}_2 \ltimes \big((S_n \ltimes \mathbb{Z}_0^n) \rtimes \mathrm{I}_n\big).$$

Thus we have shown all the statements. □

Example 2.4 The following are concrete examples of so-called diagram automorphisms relative to the Dynkin diagram of type $\mathrm{A}_4^{(1)}$ or $\mathrm{A}_5^{(1)}$.

(1) Let φ be the automorphism of $\mathrm{A}_4^{(1)}$ defined by $\varphi(\delta) = \delta$ and

$$\epsilon_1 - \epsilon_2 \mapsto \epsilon_2 - \epsilon_3, \quad \epsilon_2 - \epsilon_3 \mapsto \epsilon_3 - \epsilon_4, \quad \epsilon_3 - \epsilon_4 \mapsto \epsilon_4 - \epsilon_5 \quad \text{and} \quad \epsilon_4 - \epsilon_5 \mapsto \epsilon_5 - \epsilon_1 + \delta.$$

Then, we have $\epsilon_1 - \epsilon_3 \mapsto \epsilon_2 - \epsilon_4$, $\epsilon_1 - \epsilon_4 \mapsto \epsilon_2 - \epsilon_5$ and $\epsilon_1 - \epsilon_5 \mapsto \epsilon_2 - \epsilon_1 + \delta$, and hence,

$$\varphi = S = (1, v, 1, \overline{(0,0,0,0,1)}), \text{ where } v = (1,2,3,4,5).$$

Let φ' be the automorphism of $\mathrm{A}_4^{(1)}$ defined by $\varphi'(\delta) = \delta$ and

$$\epsilon_1 - \epsilon_2 \leftrightarrow \epsilon_4 - \epsilon_5, \quad \epsilon_2 - \epsilon_3 \leftrightarrow \epsilon_3 - \epsilon_4, \quad \text{and} \quad \epsilon_5 - \epsilon_1 + \delta \mapsto \epsilon_5 - \epsilon_1 + \delta.$$

Then, we have $\epsilon_1 - \epsilon_3 \mapsto \epsilon_3 - \epsilon_5$, $\epsilon_1 - \epsilon_4 \mapsto \epsilon_2 - \epsilon_5$ and $\epsilon_1 - \epsilon_5 \mapsto \epsilon_1 - \epsilon_5$, and hence,

$$\varphi' = T = (-1, w, 1, \bar{\mathbf{0}}), \text{ where } w = (1,5)(2,4).$$

(2) Similarly, let φ be the automorphism of $A_5^{(1)}$ defined by $\varphi(\delta) = \delta$ and

$$\epsilon_i - \epsilon_{i+1} \mapsto \epsilon_{i+1} - \epsilon_{i+2} \quad (i = 1, 2, 3, 4) \quad \text{and} \quad \epsilon_5 - \epsilon_6 \mapsto \epsilon_6 - \epsilon_1 + \delta.$$

Then, we get

$$\varphi = S = (1, w, 1, \overline{(0, 0, 0, 0, 0, 1)}), \text{ where } v = (1, 2, 3, 4, 5, 6).$$

Let φ' be the automorphism of $A_5^{(1)}$ defined by $\varphi'(\delta) = \delta$ and

$$\epsilon_1 - \epsilon_2 \leftrightarrow \epsilon_5 - \epsilon_6, \ \epsilon_2 - \epsilon_3 \leftrightarrow \epsilon_4 - \epsilon_5, \ \epsilon_3 - \epsilon_4 \mapsto \epsilon_3 - \epsilon_4, \quad \text{and} \quad \epsilon_6 - \epsilon_1 + \delta \mapsto \epsilon_6 - \epsilon_1 + \delta.$$

Then, we have $\epsilon_1 - \epsilon_3 \mapsto \epsilon_4 - \epsilon_6$, $\epsilon_1 - \epsilon_4 \mapsto \epsilon_3 - \epsilon_6$, $\epsilon_1 - \epsilon_5 \mapsto \epsilon_2 - \epsilon_6$ and $\epsilon_1 - \epsilon_6 \mapsto \epsilon_1 - \epsilon_6$, and hence,

$$\varphi' = T = (-1, w, 1, \bar{0}), \text{ where } w = (1, 6)(2, 5)(3, 4).$$

(3) In case of type $A_1^{(1)}$, we see $T = 1_V$ and, so we need to say $I_2 = \langle S \rangle = \mathbb{Z}_2$.

(4) In the case when $\mathfrak{I} = \mathbb{Z}$, which is a countable set, we let R be of type $A_{\mathfrak{I}-1}$ or of type $A_{\mathfrak{I}-1}^{(1)}$. Then, there are at least two typical outer automorphisms. That is, we have

$$\begin{aligned} &V = \oplus_{i \in \mathbb{Z} \setminus \{1\}} \mathbb{Q}(\epsilon_1 - \epsilon_i) = \oplus_{i \in \mathbb{Z}} \mathbb{Q}(\epsilon_i - \epsilon_{i+1}) \subset \hat{V} = V \oplus \mathbb{Q}\delta, \\ &\varphi : \epsilon_{i-1} - \epsilon_i \mapsto \epsilon_i - \epsilon_{i+1}, \\ &\varphi' : \epsilon_i - \epsilon_{i+1} \mapsto \epsilon_{-i} - \epsilon_{-i+1} \quad \text{or} \quad \epsilon_i - \epsilon_{i+1} \mapsto \epsilon_{-i-1} - \epsilon_{-i}. \end{aligned}$$

Then, the subgroup generated by φ and φ' is isomorphic to an infinite dihedral group I_∞.

(5) We suppose the same situation as in (4). Let us take and fix an integer $p \geq 2$. For $0 \leq i \leq p-1$, we define

$$\begin{aligned} v_i &= (\ldots, i-3p, i-2p, i-p, i, i+p, i+2p, i+3p, \ldots) \\ w_i &= (i-p, i+p)(i-2p, i+2p)(i-3p, i+3p)\cdots. \end{aligned}$$

Then, the subgroup H of $\mathrm{Aut}R$ generated by v_i and w_i for all $i = 0, 1, \ldots, p-1$ is isomorphic to

$$I_\infty^p = \underbrace{I_\infty \times \cdots \times I_\infty}_{p} = \langle v_0, w_0 \rangle \times \cdots \times \langle v_{p-1}, w_{p-1} \rangle.$$

(6) Also in the same situation as in (4), it is true that every finite group can be found in $\mathrm{Aut}\, R$ and in $\mathrm{Out}\, R$ at the same time as subgroups.

(7) Let R be an affine root system of type $\mathrm{A}_1^{(1)}$. We take $\varphi = (1, \mathrm{id}, 1, \overline{(1)})$, satisfying $\alpha_1 \mapsto \alpha_1 + \delta$ and $\delta \mapsto \delta$, and we take $\varphi' = (1, (1,2), 1, \overline{(-2)})$, satisfying $\alpha_1 \mapsto -\alpha_1 - 2\delta$ and $\delta \mapsto \delta$. Then, φ and φ' generate an infinite dihedral subgroup of $\mathrm{Aut}\, R$. Furthermore, we take $\varphi'' = (1, \mathrm{id}, -1, \bar{0})$, satisfying $\alpha_1 \mapsto \alpha_1$ and $\delta \mapsto -\delta$. Then, φ and φ'' also generate an infinite dihedral subgroup of $\mathrm{Aut}\, R$.

Remark 2.5 (1) One can determine the automorphisms of a locally finite root sysytem of type $\mathrm{A}_{\mathfrak{I}-1}$ by the same way as in the proof of Theorem 2.2. Namely, let φ be an automorphism of the root system

$$R = \{\epsilon_i - \epsilon_j \mid i \neq j \in \mathfrak{I}\}.$$

Then there exists some permutation $\sigma \in S_{\mathfrak{I}}$ such that $\varphi(\epsilon_i - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)}$ for all $i \neq j \in \mathfrak{I}$ or such that $\epsilon_{\sigma(j)} - \epsilon_{\sigma(i)}$ for all $i \neq j \in \mathfrak{I}$. Hence, $\mathrm{Aut}\, R \cong S_{\mathfrak{I}} \times \{\pm 1\}$, and the Weyl group of R is $S_{(\mathfrak{I})}$. These results are known in [5].

(2) Let R be a locally affine root system in $\hat{V}$ as before. Put $\bar{V} = \hat{V}/\mathbb{Q}\delta$, and $\omega : \hat{V} \to \bar{V}$ be a canonical linear map. Then, $\bar{R} = \omega(R)$ is a locally finite root system in $\bar{V}$. The map ω induces a group homomorphism, again called ω, of $\mathrm{Aut}\, R$ to $\mathrm{Aut}\, \bar{R}$. It is easily confirmed that this group homomorphism ω is surjective. Set $\mathrm{Aut}_0 R = \mathrm{Ker}\omega$. Therefore, we obtain the exact sequence

$$1 \longrightarrow \mathrm{Aut}_0 R \longrightarrow \mathrm{Aut}\, R \longrightarrow \mathrm{Aut}\, \bar{R} \longrightarrow 1.$$

We notice that this exact sequence is split. We also get the following exact sequence:

$$1 \longrightarrow W_0(R) \longrightarrow W(R) \longrightarrow W(\bar{R}) \longrightarrow 1,$$

where $W_0(R) = W(R) \cap \mathrm{Aut}_0 R$. Therefore, we obtain the following diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & W_0(R) & \to & W(R) & \to & W(\bar{R}) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathrm{Aut}_0 R & \to & \mathrm{Aut}\, R & \to & \mathrm{Aut}\, \bar{R} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathrm{Out}_0 R & \to & \mathrm{Out}\, R & \to & \mathrm{Out}\, \bar{R} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

We have $\mathrm{Out}_0 R = (\mathrm{Aut}_0 R)/W_0(R) \cong (\mathrm{Aut}_0 R)\, W(R)/W(R)$, and so

$$W(R) = W(\bar{R}) \ltimes W_0(R), \quad \mathrm{Aut}\, R = (\mathrm{Aut}\, \bar{R}) \ltimes (\mathrm{Aut}_0 R) \quad \text{and}$$

$$\mathrm{Out}\, R = (\mathrm{Out}\, \bar{R}) \ltimes (\mathrm{Out}_0 R).$$

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THE RELATION BETWEEN BASE CURVE OF THE RULED SURFACE AND ITS ADJOINT CURVE

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Abstract

The ruled surfaces have been investigated with the adjoint curve of the base curve in the Euclidean 3- space. We obtain the distribution parameters.In addition, some results and theorems are presented with special cases.We obtained some characterizations on the ruled surfaces by using curvatures of the adjoint curve.

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1. Introduction

Recently, the theory of surfaces and their transformations has been studied extensively in differential geometry.Ruled surfaces were investigated by G.Monge, who established the partial differential equation satisfiess by all ruled surfaces. Thus, ruled surfaces are formed by one-parameter set of lines and have been investigated by V.Hlavaty [10] and J. Hoschek [12]. Furthermore, the theory of ruled surfaces has been applied to kinematics. Especially, the study of one-parameter closed motions became an interesting subject in kinematics after the work of J. Steiner [34] and A. Holditch [11].

During the second half the nineteenth century there were appeared many publications about Steiner and Holditch theorems, for example: A. B. Kempe [15], [16], [17] and [18], C. Leudesdorf [22] and , E. B. Elliot [6], H. Frank [7], H.R. M üller [23] and [24], H. Potmann [30] and [31], O. Bottema [5], R. Thüring [37] and J.Tölke [36] etc.

The study of some classes of surfaces with special properties in E^3 such as developable, minimal is one of the principal aims of the classical differential geometry.There are many studies that interested with many properties of ruled surfaces in Euclidean space and some characterizations. Recently, O. Giering [8], C. Thas [37],Yusuf Yaylı [39],[40],[29],N. Kuruoglu and S. Keleş [19]etc. have used the this invariants in their investigations concerning the generalization of some theorems of the theory ruled surfaces.

Let E^3 be a 3-dimensional Euclidean space provided with the metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2 \tag{1.1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Let $\alpha : I \subset \mathbb{R} \to E^3$ is a unit speed curve with $\tau_\alpha \neq 0$. Denote by $\{T_\alpha, N_\alpha, B_\alpha\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E^3.The Frenet equations are

$$\begin{bmatrix} T'_\alpha(s) \\ N'_\alpha(s) \\ B'_\alpha(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{bmatrix} \begin{bmatrix} T_\alpha(s) \\ N_\alpha(s) \\ B_\alpha(s) \end{bmatrix} \tag{1.2}$$

Adjoint curve of α is defined by

$$\beta(s) = \int_{s_0}^{s} B_\alpha(s) ds \tag{1.3}$$

Let $\beta : I \subset \mathbb{R} \to E^3$ is a unit speed curve with $\tau_\beta > 0$. Denote by $\{T_\beta, N_\beta, B_\beta\}$ the moving Frenet frame along the curve $\beta(s)$ in the space E^3.Differentiating the equation (1.3),we have

$$\begin{aligned} T_\beta(s) &= B_\alpha(s) \\ N_\beta(s) &= -N_\alpha(s) \\ B_\beta(s) &= T_\alpha(s) \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \kappa_\beta(s) &= \tau_\alpha(s) \\ \tau_\beta(s) &= \kappa_\alpha(s) \end{aligned} \tag{1.5}$$

The ruled surface $\phi(s,v)$ is generated by a one-parameter family of straight lines and it possesses a parametric representation

$$\begin{aligned} \phi: \quad I \times \mathbb{R} &\to \mathbb{R}^3 \\ \phi(s,v) &= \alpha(s) + vX(s) \end{aligned} \tag{1.6}$$

where $\alpha(s)$ is called the base curve and $X(s)$ is the unit represents a space curve which representing the direction of straight line [27].
If there exists a common perpendicular to two constructive rulings in the ruled surface, then the foot of the common perpendicular on the main rulings is called a central point. The locus of the central point is called striction curve . The parametrization of the striction curve on the ruled surface (1.5) is given by

$$\bar{\alpha}(s) = \alpha(s) - \frac{\langle \alpha'(s), X'(s) \rangle}{\|X'(s)\|^2} X(s) \tag{1.7}$$

On the other hand, the Gaussian curvature K, the mean curvature H and the distribution parameter P_x are given by, respectively

$$\begin{aligned} K &= \frac{ln - m^2}{EG - F^2} \\ H &= \frac{1}{2} \frac{Gl + En - 2Fm}{EG - F^2} \\ P_x &= \frac{\det(T_\alpha, X, D_{T_\alpha} X)}{\langle D_{T_\alpha} X, D_{T_\alpha} X \rangle} \end{aligned} \tag{1.8}$$

where D is the Levi-Civita connection on $\mathbb{R}^3$ [27].

Theorem 1. *A ruled surface is a developable surface if and only if the distribution parameter of the ruled surface is zero* [27].

2. One Parameter Spatial Motion in $\mathbb{E}^3$

Let α be a curve with arclength parameter s, $\{T_\alpha, N_\alpha, B_\alpha\}$ be the Frenet vector fields of α and β be the adjoint curve α.
The moving frames $\{O; T_\alpha, N_\alpha, B_\alpha\}$ and $\{O_1, e_1, e_2, e_3\}$ are orthogonal frames, which represent respectively the moving space H and the fixed space H_1. Let us express the displacements $H\backslash H_1$ of H with respect to H_1. During the one parameter spatial motion $H\backslash H_1$, each fixed line X of the moving space H generates, in generally, a ruled surface in the fixed space H_1.

Let $\phi(s,v)$ be a ruled surface given by the parametrization (1.6) and X be a fixed unit vector. Thus

$$X \in Sp\{T_\alpha, N_\alpha, B_\alpha\} \text{ and } X = x_1 T_\alpha + x_2 N_\alpha + x_3 B_\alpha \tag{2.1}$$

is such that

$$\langle X, X\rangle = 1.$$

We can obtain the distribution parameter of the ruled surface generated by line X of the moving space H. From (1.3)

$$X = x_3 T_\beta - x_2 N_\beta + x_1 B_\beta, \quad x_1^2 + x_2^2 + x_3^2 = 1 \tag{2.2}$$

Differentiating (2.2) with respect to s, we get

$$X' = x_3 T_\beta' - x_2 N_\beta' + x_1 B_\beta' \tag{2.3}$$

By using the Frenet frame of the adjoint curve β in (2.3),we obtain

$$D_{T_\alpha} X = x_2 \kappa_\beta T_\beta + (x_3 \kappa_\beta - x_1 \tau_\beta) N_\beta - x_2 \tau_\beta B_\beta \tag{2.4}$$

From (1.7) we get

$$P_x = \frac{\left(x_1 x_3 - x_2^2\right) \tau_\beta - x_3^2 \kappa_\beta}{x_2^2 \kappa_\beta^2 + \left(x_3 \kappa_\beta - x_1 \tau_\beta\right)^2 + x_2^2 \tau_\beta^2} \tag{2.5}$$

Theorem 2. *Let $\phi(s,v)$ be a ruled surface given by the parametrization (1.5) . $\phi(s,v)$ is developable if and only if the adjoint curve β is a helix such that the curvatures of β, κ_β and τ_β satisfy*

$$\frac{\kappa_\beta}{\tau_\beta} = \frac{x_1 x_3 - x_2^2}{x_3^2} \tag{2.6}$$

Proof. Let $\phi(s,v)$ be a ruled surface. By using (2.5) and theorem 1 ,

$$\left(x_1 x_3 - x_2^2\right) \tau_\beta - x_3^2 \kappa_\beta = 0 \tag{2.7}$$

is obtained. Then

$$\frac{\kappa_\beta}{\tau_\beta} = \frac{x_1 x_3 - x_2^2}{x_3^2} = constant \tag{2.8}$$

So, β is a helix.This completes the proof. □

3. Special Cases

Let $\phi(s,v)$ be a ruled surface given by the parametrization (1.5) , X be the director vector of the base curve α and β be adjoint curve of the base curve α.

3.1. **The Case $X = T_\beta$.** In this case, $x_1 = x_2 = 0,\ x_3 = 1$.Thus from (2.5)

$$P_x = \frac{-1}{\kappa_\beta} \tag{3.1}$$

Corollary 1. *According to the adjoint curve β of the base curve α, there is no developable ruled surface generated by the T_β line in $\mathbb{E}^3$.*

It is obvious from (3.1) .

3.2. **The Case $X = N_\beta$.** In this case, $x_1 = x_3 = 0,\ x_2 = -1$.Thus from (2.5)

$$P_x = \frac{-\tau_\beta}{\kappa_\beta^2 + \tau_\beta^2} \tag{3.2}$$

Corollary 2. *Let $\phi(s,v)$ be a ruled surface is generated by the N_β line. $\phi(s,v)$ is developable if and only if β is a planar curve.*

Proof. Let $\phi(s,v)$ be a ruled surface. By using (3.2) and theorem 1 we obtain $\tau_\beta = 0$.This completes the proof. □

3.3. **The Case $X = B_\beta$.** In this case, $x_1 = 1, x_2 = x_3 = 0$.Thus from (2.5) , it is obvious that If $\tau_\beta \neq 0$,then $P_x = 0$.

3.4. **The Case $X \in Sp\{T_\beta, N_\beta\}$.** In this case, $x_1 = 0, x_2 = -1, x_3 = 1$.From (2.5), the distribution parameter of the ruled surface is given by

$$P_x = \frac{-(\tau_\beta - \kappa\beta)}{2\kappa_\beta^2 + \tau_\beta^2} \tag{3.3}$$

Corollary 3. *Let $\phi(s,v)$ be a ruled surface is generated by the $X \in Sp\{T_\beta, N_\beta\}$ line. $\phi(s,v)$ is developable if and only if α is helix.*

Proof. Let $\phi(s,v)$ be a ruled surface. By using (3.3) and theorem 1 we obtain $\tau_\beta = -\kappa_\beta$.This completes the proof. □

3.5. **The Case $X \in Sp\{T_\beta, B_\beta\}$.** In this case, $x_2 = 0, x_1 = x_3 = 1$.From (2.5), the distribution parameter of the ruled surface is given by

$$P_x = \frac{-1}{(\kappa_\beta - \tau_\beta)} \tag{3.4}$$

Corollary 4. *Let $\phi(s,v)$ be a ruled surface is generated by the $X \in Sp\{T_\beta, B_\beta\}$ line.There is no developable ruled surface in this case.*

It can be easily seen from (3.4) .

3.6. **The Case** $X \in Sp\{N_\beta, B_\beta\}$. In this case, $x_1 = 1, x_2 = -1, x_3 = 0$.From (2.5), the distribution parameter of the ruled surface is given by

$$P_x = \frac{-\tau_\beta}{\kappa_\beta^2}. \tag{3.5}$$

Corollary 5. *Let $\phi(s,v)$ be a ruled surface is generated by the $X \in Sp\{N_\beta, B_\beta\}$ line. $\phi(s,v)$ is developable if and only if β is a planar curve.*

Proof. Let $\phi(s,v)$ be a ruled surface given by the parametrization (3.5) .If the ruled surface is developable, from theorem 1 $P_x = 0$.This completes the proof. □

By using (1.7) we compute the mean curvatures and Gauss curvatures of the ruled surfaces generated by vectors $\{T_\beta, N_\beta, B_\beta\}$.

Proposition 1. *Let $\phi(s,v)$ be a ruled surface generated by the tangant line T_β of the adjoint curve β.From (1.7) , Gauss curvature and the mean curvature are obtained as follows:*

$$K_{T_\beta} = \frac{\kappa_\beta^2}{\left(1 + v^2\kappa_\beta^2\right)^2} \tag{3.6}$$

$$H_{T_\beta} = \frac{1}{2}\frac{v\kappa_\beta' - \tau_\beta - v^2\kappa_\beta^2\tau_\beta}{\left(1 + v^2\kappa_\beta^2\right)^{\frac{3}{2}}} \tag{3.7}$$

Corollary 6. *Let $\phi(s,v)$ be a ruled surface generated by the tangant line T_β of the adjoint curve β. $\phi(s,v)$ is a minimal ruled surface if and only if*

$$\kappa_\beta' = \frac{v^2\kappa_\beta^2 + 1}{v}\tau_\beta \tag{3.8}$$

Proof. Let $\phi(s,v)$ be a minimal ruled surface. (3.7)

$$v\kappa_\beta' - \tau_\beta - v^2\kappa_\beta^2 = 0$$

This completes the proof. □

Proposition 2. *Let $\phi(s,v)$ be a ruled surface generated by the normal line N_β of the adjoint curve β.From (1.7) , Gauss curvature and the mean curvature are obtained as follows:*

$$K_{T_\beta} = \frac{-\kappa_\beta^2}{\left[\left(1 + v\tau_\beta\right)^2 + v^2\kappa_\beta^2\right]^2} \tag{3.9}$$

$$H_{T_\beta} = \frac{1}{2}\frac{-v\kappa_\beta' - v^2\kappa_\beta'\tau_\beta + v\kappa_\beta\tau_\beta'}{\left[\left(1 + v\tau_\beta\right)^2 + v^2\kappa_\beta^2\right]^{\frac{3}{2}}} \tag{3.10}$$

Corollary 7. *Let $\phi(s,v)$ be a ruled surface generated by the normal line N_β of the adjoint curve β.$\phi(s,v)$ is a minimal ruled surface if and only if*

$$\frac{\kappa_\beta'}{\tau_\beta'} = \frac{\kappa_\beta}{1+v\tau_\beta} \tag{3.11}$$

Proof. Let $\phi(s,v)$ be a minimal ruled surface. From (3.10)

$$-v\kappa_\beta' - v^2\kappa_\beta'\tau_\beta + v\kappa_\beta\tau_\beta' = 0$$

we get

$$\frac{\kappa_\beta'}{\tau_\beta'} = \frac{\kappa_\beta}{1+v\tau_\beta}$$

Proposition 3. *Let $\phi(s,v)$ be a ruled surface generated by the binormal line B_β of the adjoint curve β.From (1.7) , Gauss curvature and the mean curvature are obtained as follows:*

$$K_{N_\beta} = 0 \tag{3.12}$$

$$H_{N_\beta} = \frac{-1}{2}\frac{\kappa_\beta}{v\tau_\beta} \tag{3.13}$$

Corollary 8. *Let $\phi(s,v)$ be a ruled surface generated by the binormal line N_β of the adjoint curve β.$\phi(s,v)$ is a minimal ruled surface if and only if*

$$\tau_\beta \neq 0 \text{ and } \kappa_\beta = 0 \tag{3.14}$$

It is obvious from (3.13).

Proposition 4. *Let $\phi(s,v)$ be a ruled surface given by the parametrization (1.5) .If the base curve α of $\phi(s,v)$ is also striction curve, then $x_2 = 0$ or the adjoint curve β of the base curve α is a planar curve.*

Proof. Let the base curve α be the striction curve.Thus, from (1.6),

$$\langle T_\alpha, D_{T_\alpha}X\rangle = 0$$

By using (1.3) and (2.4) we obtain

$$-x_2\tau_\beta = 0$$

This completes the proof. □

Proposition 5. *Let $\phi(s,v)$ be a ruled surface given by the parametrization (1.5).If the base curve α of $\phi(s,v)$ is also asymtotic curve, then*

$$-\tau_\beta = 0 \text{ or } (x_3 - vx_2x_3\tau_\beta - vx_1x_2\kappa_\beta) = 0 \tag{3.15}$$

Proof. We assume that the base curve of the surface $\phi(s,v)$ is the asymtotic curve. In that case,

$$\langle \alpha'', N\rangle = 0 \tag{3.16}$$

where

$$N = \frac{\phi_s \wedge \phi_v}{\|\phi_s \wedge \phi_v\|} \tag{3.17}$$

is a unit normal vector of the ruled surface ϕ.

$$N = \frac{1}{\|\phi_s \wedge \phi_v\|} \begin{bmatrix} \left(vx_1x_3\kappa_\beta - vx_1^2\tau_\beta + x_2 - vx_2^2\tau_\beta\right) T_\beta \\ \left(+x_3 - vx_2x_3\tau_\beta - vx_1x_2\kappa_\beta\right) N_\beta \\ +\left(-vx_2^2\kappa_\beta - vx_3^2\kappa_\beta + vx_1x_3\tau_\beta\right) B_\beta \end{bmatrix} \tag{3.18}$$

From (1.1), (1.3) and (1.4),we get

$$-\tau_\beta \left(x_3 - vx_2x_3\tau_\beta - vx_1x_2\kappa_\beta\right) = 0$$

This completes the proof. □

Proposition 6. *Let $\phi(s,v)$ be a ruled surface given by the parametrization (1.5). We obtain the following results for the ruled surfaces.*

(i)The s-parameter curve of $\phi(s,v)$ is also an asymtotic curve if and only if

$$\begin{aligned}(\kappa_\beta(2v^2x_1^2x_3 + v^2x_2^2x_3 - v^2x_3^2) + (2v^2x_1x_3^3 - v^2x_1x_2^2 - v^2x_1^3)\tau_\beta + 2vx_1x_2 + vx_3)\kappa_\beta\tau_\beta \\ (-x_3 + (2vx_2x_3 - vx_1)\tau_\beta - v^2x_3(1-x_3^2)\tau_\beta^2)\tau_\beta + v^2x_2(\kappa_\beta\tau_\beta' - \kappa_\beta'\tau_\beta) \\ + v(1-x_1^2)\kappa_\beta' - v(x_1x_3 + x_2)\tau_\beta' - v^2x_1x_2^2\kappa_\beta^3\end{aligned}$$

(ii)The v-parameter curve of $\phi(s,v)$ is also an asymtotic curve if and only if

$$(1-x_1^2)\kappa_\beta - (x_1x_3 + x_2)\tau_\beta = 0 \tag{3.19}$$

Proof. (i) If the s-parameter curve of $\phi(s,v)$ is also an asymtotic curve, then

$$\langle \phi_{ss}, N\rangle = 0. \tag{3.20}$$

Differentiating ϕ_s with respect to s and (3.18),we obtain (3.19).
(ii) If the v-parameter curve of $\phi(s,v)$ is also an asymtotic curve, then

$$\langle \phi_{vv}, N\rangle = 0. \tag{3.21}$$

Differentiating ϕ_v with respect to v and 3.18,we obtain (3.21) . □

Proposition 7. *Let $\phi(s,v)$ be a ruled surface given by the parametrization (1.5) . If the base curve of $\phi(s,v)$ is also a jeodesic curve, then the curvature functions κ_β and τ_β of the adjoint curve β of the base curve α satisfy the following equation:*

$$\begin{aligned}-\tau_\beta\left(-vx_2^2\kappa_\beta - vx_3^2\kappa_\beta + vx_1x_3\tau_\beta\right) = 0 \\ -\tau_\beta\left(-vx_1x_3\kappa_\beta - vx_1^2\tau_\beta + x_2 - vx_2^2\tau_\beta\right) = 0\end{aligned} \tag{3.22}$$

Proof. Let the base curve of $\phi(s,v)$is also a jeodesic curve, then

$$\alpha'' \times N = 0 \tag{3.23}$$

Applying (1.3) and (3.18), we complete the proof. □

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